9) Find all critical points of $f(x,y) = 2x^3 + 2y^3 + 3y^2 - 54x - 12y + 1$, and classify each one as a local maximum, a local minimum, or a saddle point. Show all work, as we have done in class. You do <u>not</u> have to find the corresponding function values. (27 points)

<u>Step 1</u>: Find any critical points (CPs).

$$f_x(x,y) = 6x^2 - 54 \qquad \left(\text{never DNE; is continuous on } \mathbf{R}^2\right)$$
$$f_y(x,y) = 6y^2 + 6y - 12 \qquad \left(\text{never DNE; is continuous on } \mathbf{R}^2\right)$$

Solve the system: $\begin{cases} 6x^2 - 54 = 0 \\ 6y^2 + 6y - 12 = 0 \end{cases}$

If we divide both sides of both equations by 6, we obtain the simpler system:

$$\begin{cases} x^2 - 9 = 0 \\ y^2 + y - 2 = 0 \end{cases}$$

If we solve the first equation for x, we get:

$$x^2 - 9 = 0$$
$$x^2 = 9$$
$$x = \pm 3$$

If we solve the second equation for y, we get:

$$y^2 + y - 2 = 0$$
$$(y+2)(y-1) = 0$$

$$y + 2 = 0 \quad \text{or} \quad y - 1 = 0$$
$$y = -2 \qquad\qquad y = 1$$

Each of our x values can be paired with each of our y values.

Our critical points (CPs) are:

$(-3,-2)$, $(-3,1)$, $(3,-2)$, and $(3,1)$.

(Note that they are in the domain of f.)

<u>Step 2</u>: Find f_{xx} and D.

Remember,

$$f_x(x,y) = 6x^2 - 54$$
$$f_y(x,y) = 6y^2 + 6y - 12$$

$$f_{xx}(x,y) = 12x$$
$$f_{xy}(x,y) = 0 \qquad \left(= f_{yx}(x,y);\ \text{note that "0" is continuous.}\right)$$
$$f_{yy}(x,y) = 12y + 6$$

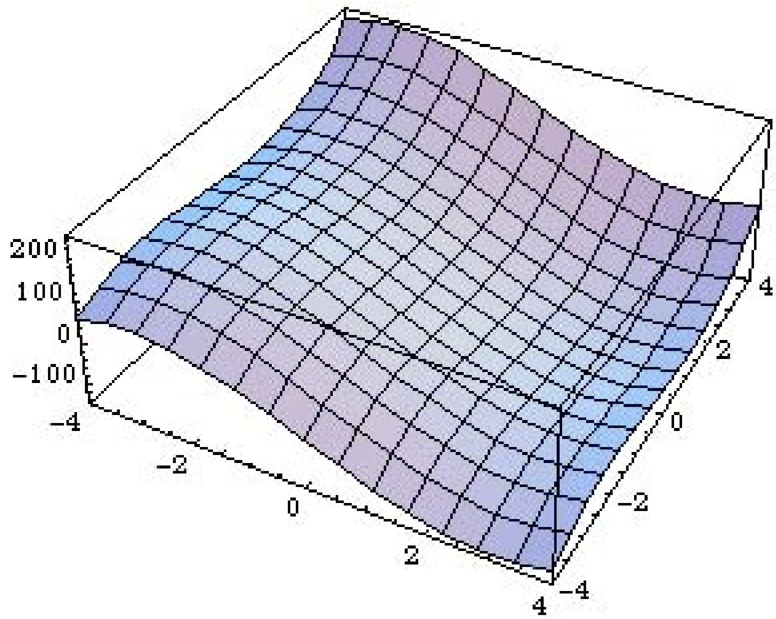
$$\begin{aligned} D &= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \\ &= \begin{vmatrix} 12x & 0 \\ 0 & 12y+6 \end{vmatrix} \\ &= (12x)(12y+6) - 0 \\ &= (12x)[6(2y+1)] \\ &= 72x(2y+1) \end{aligned}$$

<u>Step 3</u>: Classify the critical points (CPs).

Critical Point	$D = 72x(2y+1)$	$f_{xx} = 12x$	Classification
$(-3,-2)$	$D = (72)(-3)[2(-2)+1]$ $= (+)(-)[-]$ $= +$ (specifically, 648)	$f_{xx} = 12(-3)$ $= -36$ < 0 ("−") Think: "Concave down" $(\cap)$	**Local Maximum**
$(-3,1)$	$D = (72)(-3)[2(1)+1]$ $= (+)(-)[+]$ $= -$ (specifically, -648)	irrelevant	**Saddle Point**
$(3,-2)$	$D = (72)(3)[2(-2)+1]$ $= (+)(+)[-]$ $= -$ (specifically, -648)	irrelevant	**Saddle Point**
$(3,1)$	$D = (72)(3)[2(1)+1]$ $= (+)(+)[+]$ $= +$ (specifically, 648)	$f_{xx} = 12(3)$ $= 36$ > 0 ("+") Think: "Concave up" $(\cup)$	**Local Minimum**

Note: The corresponding points of interest on the graph are:
$(-3,-2,129)$, $(-3,1,102)$, $(3,-2,-87)$, and $(3,1,-114)$.

Here is a graph:



10) Find the absolute maximum and absolute minimum of $f(x,y,z)=3x-2y+5z$ subject to the constraint $9x^2+y^2+\frac{15}{2}z^2=1$ using the method of Lagrange multipliers. Your answers will be ordered triples in the domain of f; label which one corresponds to the absolute maximum and which one corresponds to the absolute minimum. Give exact, simplified answers, and rationalize denominators. (25 points)

Take the constraint equation, and isolate 0 on one side:

$$9x^2+y^2+\frac{15}{2}z^2=1 \quad \text{(Its graph is an ellipse centered at the origin.)}$$

$$\underbrace{9x^2+y^2+\frac{15}{2}z^2-1}_{g(x,y,z)}=0$$

Solve $\begin{cases} \nabla f(x,y,z)=\lambda \nabla g(x,y,z) \\ g(x,y,z)=0 \end{cases}$

$$\nabla f(x,y,z)=\lambda \nabla g(x,y,z)$$
$$\left\langle f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z)\right\rangle=\lambda\left\langle g_x(x,y,z),\ g_y(x,y,z),\ g_z(x,y,z)\right\rangle$$
$$\langle 3,\ -2,\ 5\rangle=\lambda\langle 18x,\ 2y,\ 15z\rangle$$

$$\text{Solve} \begin{cases} 3 = \lambda(18x) & (\text{Eq.1}) \\ -2 = \lambda(2y) & (\text{Eq.2}) \\ 5 = \lambda(15z) & (\text{Eq.3}) \\ 9x^2 + y^2 + \frac{15}{2}z^2 - 1 = 0 & (\text{Eq.C}) \end{cases}$$

Solve the first three equations for λ. Observe that $x = 0$, $y = 0$, and $z = 0$ cannot solve the first three equations, so we may assume that x, y, and z are nonzero.

$$(\text{Eq.1}) \Rightarrow \lambda = \frac{3}{18x} \Rightarrow \lambda = \frac{1}{6x}$$

$$(\text{Eq.2}) \Rightarrow \lambda = \frac{-2}{2y} \Rightarrow \lambda = -\frac{1}{y}$$

$$(\text{Eq.3}) \Rightarrow \lambda = \frac{5}{15z} \Rightarrow \lambda = \frac{1}{3z}$$

Equate our expressions for λ.

$$\lambda = \frac{1}{6x} = -\frac{1}{y} = \frac{1}{3z}$$

Equate the reciprocals of the last three expressions:

$$6x = -y = 3z$$

We may then solve for y and z in terms of x:

$$6x = -y \Rightarrow y = -6x$$

$$6x = 3z \Rightarrow z = 2x$$

Substitute $y=-6x$ and $z=2x$ into (Eq.C).

$$9x^2+y^2+\frac{15}{2}z^2-1=0$$
$$9x^2+(-6x)^2+\frac{15}{2}(2x)^2-1=0$$
$$9x^2+36x^2+\frac{15}{2}(4x^2)-1=0$$
$$9x^2+36x^2+30x^2-1=0$$
$$75x^2=1$$
$$x^2=\frac{1}{75}$$
$$x=\pm\sqrt{\frac{1}{75}}$$
$$x=\pm\frac{1}{5\sqrt{3}} \quad \text{or} \quad \pm\frac{\sqrt{3}}{15}$$

Find our candidate points. Remember: $y=-6x$ and $z=2x$.

$$x=\frac{\sqrt{3}}{15} \quad \Rightarrow \quad y=-6\left(\frac{\sqrt{3}}{15}\right)=-\frac{2\sqrt{3}}{5} \quad \text{and} \quad z=2\left(\frac{\sqrt{3}}{15}\right)=\frac{2\sqrt{3}}{15}.$$

$$x=-\frac{\sqrt{3}}{15} \quad \Rightarrow \quad y=-6\left(-\frac{\sqrt{3}}{15}\right)=\frac{2\sqrt{3}}{5} \quad \text{and} \quad z=2\left(-\frac{\sqrt{3}}{15}\right)=-\frac{2\sqrt{3}}{15}.$$

Candidate point 1: $\left(\frac{\sqrt{3}}{15},-\frac{2\sqrt{3}}{5},\frac{2\sqrt{3}}{15}\right)$

Function value from $f(x,y,z)=3x-2y+5z$:

$$3\left(\frac{\sqrt{3}}{15}\right)-2\left(-\frac{2\sqrt{3}}{5}\right)+5\left(\frac{2\sqrt{3}}{15}\right)=\frac{3\sqrt{3}}{15}+\frac{4\sqrt{3}}{5}+\frac{10\sqrt{3}}{15}$$
$$=\frac{3\sqrt{3}}{15}+\frac{12\sqrt{3}}{15}+\frac{10\sqrt{3}}{15}$$
$$=\frac{25\sqrt{3}}{15}$$
$$=\frac{5\sqrt{3}}{3}$$

Candidate point 2:

$$\left(-\frac{\sqrt{3}}{15}, \frac{2\sqrt{3}}{5}, -\frac{2\sqrt{3}}{15}\right)$$

Function value from $f(x,y,z) = 3x - 2y + 5z$:

$$3\left(-\frac{\sqrt{3}}{15}\right) - 2\left(\frac{2\sqrt{3}}{5}\right) + 5\left(-\frac{2\sqrt{3}}{15}\right) = -\frac{3\sqrt{3}}{15} - \frac{4\sqrt{3}}{5} - \frac{10\sqrt{3}}{15}$$
$$= -\frac{5\sqrt{3}}{3}$$

Our candidates are both in the domain of f .

We know that the candidates for local extrema we find are absolute extrema, because there are only two, the graph of the constraint is a closed surface, and f is continuous on it. We may identify which is the absolute maximum and which is the absolute minimum by evaluating and comparing values of f at these candidate points, or (in this case) by applying a sign analysis: Observe that evaluating $3x - 2y + 5z$ at the first candidate point yields three positive terms, while evaluating it at the second candidate point yields three negative terms.

$\left(\frac{\sqrt{3}}{15}, -\frac{2\sqrt{3}}{5}, \frac{2\sqrt{3}}{15}\right)$ corresponds to the absolute maximum, and
$\left(-\frac{\sqrt{3}}{15}, \frac{2\sqrt{3}}{5}, -\frac{2\sqrt{3}}{15}\right)$ corresponds to the absolute minimum.