QUIZ 3 (SECTIONS 16.3-16.9) SOLUTIONS

MATH 252 – FALL 2007 – KUNIYUKI

SCORED OUT OF 125 POINTS ⇒ MULTIPLIED BY 0.84 ⇒ 105% POSSIBLE

1) Let $f(x,y,z)=\sqrt{3x^2y+z^3}$. Find $f_x(x,y,z)$. (4 points)

$$\begin{aligned}
f_x(x,y,z) &= D_x\left(\sqrt{3x^2y+z^3}\right) \\
&= D_x\left[\left(3x^2y+z^3\right)^{1/2}\right] \\
&= \frac{1}{2}\left(3x^2y+z^3\right)^{-1/2}\cdot D_x\left(3x^2\underbrace{y}_{\text{"\#"}}+\underbrace{z^3}_{\text{"\#"}}\right) \\
&= \frac{\cancel{1}}{\cancel{2}}\left(3x^2y+z^3\right)^{-1/2}\cdot\left[3y\left(\cancel{2}x\right)\right] \\
&= \boxed{\frac{3xy}{\sqrt{3x^2y+z^3}}}
\end{aligned}$$

2) Let $f(r,s)=\cos(rs)$. Find $f_r(r,s)$ and use that to find $f_{rs}(r,s)$. (6 points)

$$\begin{aligned}
f_r(r,s) &= D_r\left[\cos(rs)\right] \\
&= \left[-\sin(rs)\right]\left[D_r\left(r\underbrace{s}_{\text{"\#"}}\right)\right] \\
&= \left[-\sin(rs)\right]\left[s\right] \\
&= -s\sin(rs)
\end{aligned}$$

$$f_{rs}(r,s) = D_s\left[-s\sin(rs)\right]$$

We will use a Product Rule for Differentiation.

$$\begin{aligned}
&= \left[D_s(-s)\right]\left[\sin(rs)\right] + \left[-s\right]\left(D_s\left[\sin(rs)\right]\right) \\
&= \left[-1\right]\left[\sin(rs)\right] + \left[-s\right]\left[\cos(rs)\right]\left[D_s\left(\underbrace{r}_{\text{"\#"}}s\right)\right] \\
&= \left[-1\right]\left[\sin(rs)\right] + \left[-s\right]\left[\cos(rs)\right]\left[r\right] \\
&= \boxed{-\sin(rs)-rs\cos(rs)}
\end{aligned}$$

3) Assume that f is a function of x and y. Write the limit definition of $f_y(x,y)$ using the notation from class. (4 points)

$$\boxed{f_y(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}}$$

4) Let $f(x,y)=3xy^2-4y^3+5$. Use differentials to linearly approximate the change in f if (x,y) changes from $(4,-2)$ to $(3.98,-1.96)$. (12 points)

$$\begin{aligned} f_x(x,y) &= D_x\left(3x\underbrace{y^2}_{\text{"\#"}}-4\underbrace{y^3}_{\text{"\#"}}+5\right) \\ &= 3y^2 \\ f_x(4,-2) &= 3(-2)^2 \\ &= 12 \end{aligned}$$

$$\begin{aligned} dx &= \text{new } x - \text{old } x \\ &= 3.98-4 \\ &= -0.02 \end{aligned}$$

$$\begin{aligned} f_y(x,y) &= D_y\left(3\underbrace{x}_{\text{"\#"}}y^2-4y^3+5\right) \\ &= 3x(2y)-12y^2 \\ &= 6xy-12y^2 \\ f_y(4,-2) &= 6(4)(-2)-12(-2)^2 \\ &= -96 \end{aligned}$$

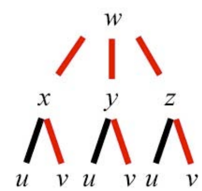
$$\begin{aligned} dy &= \text{new } y - \text{old } y \\ &= -1.96-(-2) \\ &= 0.04 \end{aligned}$$

The approximate change in f is given by:

$$\begin{aligned} df &= \left[f_x(4,-2)\right]dx + \left[f_y(4,-2)\right]dy \\ &= [12](-0.02) + [-96](0.04) \\ &= \boxed{-4.08} \end{aligned}$$

<u>Note</u>: Actual change ≈ -4.01315

5) Let f, f_1, f_2 and f_3 be differentiable functions such that $w = f(x,y,z)$, $x = f_1(u,v)$, $y = f_2(u,v)$, and $z = f_3(u,v)$. Use the Chain Rule to write an expression for $\dfrac{\partial w}{\partial v}$. (5 points)



$$\boxed{\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}}$$

6) Find $\dfrac{\partial z}{\partial x}$ if $z = f(x,y)$ is a differentiable function described implicitly by the equation $e^{xyz} = xz^4$. Use the Calculus III formula given in class. Simplify. (9 points)

First, isolate 0 on one side: $\underbrace{e^{xyz} - xz^4}_{\text{Let this be } F(x,y,z)} = 0$

Find $\dfrac{\partial z}{\partial x}$. When using the formula, treat x, y, and z as independent variables.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$= -\frac{D_x\left[e^{xyz} - x\overbrace{z^4}^{\text{"\#"}}\right]}{D_z\left[e^{xyz} - \overbrace{x}^{\text{"\#"}}z^4\right]}$$

$$= -\frac{e^{xyz}\cdot D_x\left(x\overbrace{yz}^{\text{"\#"}}\right) - z^4}{e^{xyz}\cdot D_z\left(\underbrace{xy}_{\text{"\#"}}z\right) - x\left(4z^3\right)}$$

$$= -\frac{e^{xyz}\cdot(yz) - z^4}{e^{xyz}\cdot(xy) - 4xz^3}$$

$$= \boxed{-\frac{yze^{xyz} - z^4}{xye^{xyz} - 4xz^3} \text{ or } \frac{z^4 - yze^{xyz}}{xye^{xyz} - 4xz^3} \text{ or } \frac{yze^{xyz} - z^4}{4xz^3 - xye^{xyz}}}$$

7) The temperature at any point (x, y) in the *xy*-plane is given by $f(x, y) = 2xy + x^2$ in degrees Fahrenheit. Assume x and y are measured in meters. Give units in your answers. (23 points total)

a) Find the **maximum** rate of change of temperature at the point $(3, 4)$. Approximate your final answer to three significant digits.

$$\begin{aligned}\nabla f(x, y) &= \langle f_x(x, y), f_y(x, y) \rangle \\ &= \left\langle D_x\left(2x \underbrace{y}_{\text{"\#"}} + x^2\right),\ D_y\left(2\underbrace{x}_{\text{"\#"}} y + \underbrace{x^2}_{\text{"\#"}}\right)\right\rangle \\ &= \langle 2y + 2x,\ 2x \rangle\end{aligned}$$

$$\begin{aligned}\nabla f(3, 4) &= \langle 2(4) + 2(3),\ 2(3) \rangle \\ &= \langle 14, 6 \rangle\end{aligned}$$

The length of the gradient of f at $(3, 4)$ gives the **maximum** rate of change of temperature at that point.

$$\begin{aligned}\|\nabla f(3, 4)\| &= \|\langle 14, 6 \rangle\| \\ &= 2\|\langle 7, 3 \rangle\| \\ &= 2\sqrt{(7)^2 + (3)^2} \\ &= 2\sqrt{58} \\ &\approx 15.2\end{aligned}$$

Answer: $\boxed{\text{About } 15.2 \dfrac{{}^\circ\text{F}}{\text{m}}}$

b) Find the rate of change of temperature at $(3, 4)$ in the direction of $\mathbf{i} - 3\mathbf{j}$. Approximate your final answer to three significant digits.

Let **a** be the given direction vector $\mathbf{i} - 3\mathbf{j}$, or $\langle 1, -3 \rangle$.

Find the unit vector **u** in the direction of **a**.

$$\begin{aligned}
\mathbf{u} &= \frac{\mathbf{a}}{\|\mathbf{a}\|} \\
&= \frac{\langle 1, -3 \rangle}{\|\langle 1, -3 \rangle\|} \\
&= \frac{\langle 1, -3 \rangle}{\sqrt{(1)^2 + (-3)^2}} \\
&= \frac{\langle 1, -3 \rangle}{\sqrt{10}} \quad \text{or} \quad \frac{1}{\sqrt{10}} \langle 1, -3 \rangle
\end{aligned}$$

The directional derivative at $(3, 4)$ in the direction of **u** is:

$$\begin{aligned}
D_{\mathbf{u}} f(3, 4) &= \nabla f(3, 4) \bullet \mathbf{u} \\
&= \langle 14, 6 \rangle \bullet \left(\frac{1}{\sqrt{10}} \langle 1, -3 \rangle \right) \\
&= \frac{1}{\sqrt{10}} \left(\langle 14, 6 \rangle \bullet \langle 1, -3 \rangle \right) \\
&= \frac{1}{\sqrt{10}} (-4) \\
&= \frac{-4}{\sqrt{10}} \quad \text{or} \quad -\frac{4\sqrt{10}}{10} \\
&= -\frac{2\sqrt{10}}{5} \\
&\approx -1.26
\end{aligned}$$

Answer: $\boxed{\text{About } -1.26 \, \frac{{}^{\circ}\text{F}}{\text{m}}}$

c) Find a non-**0** direction vector **v** such that the rate of change of temperature at $(3, 4)$ in the direction of **v** is 0 [units].

We want a tangent vector to the level curve of f through the point $(3, 4)$.

Any non-**0** vector orthogonal to $\nabla f(3, 4)$, which is $\langle 14, 6\rangle$, will do.

Observe that $\langle 6, -14\rangle \perp \langle 14, 6\rangle$, since $\langle 6, -14\rangle \bullet \langle 14, 6\rangle = 0$.

Therefore, $\langle 6\ [\mathrm{m}], -14\ [\mathrm{m}]\rangle$ or any non-**0** scalar multiple will do.
In particular, the simpler vector $\langle 3\ [\mathrm{m}], -7\ [\mathrm{m}]\rangle$ will do.

8) Find an equation for the tangent plane to the graph of the equation $5x^2 - 4y^2 + z^2 = 45$ at the point $P(-3, 2, 4)$. (10 points)

Observe that the given graph is a hyperboloid of one sheet.
(You may check that the coordinates of *P* satisfy the given equation, meaning that *P* lies on the graph of the equation.)

Isolate 0 on one side of the given equation.

$$\underbrace{5x^2 - 4y^2 + z^2 - 45}_{=F(x,y,z)} = 0$$

A normal vector for the desired tangent plane is given by $\nabla F(-3, 2, 4)$.

$$\begin{aligned}\nabla F\big|_P &= \left\langle F_x\big|_P,\ F_y\big|_P,\ F_z\big|_P\right\rangle \\ &= \langle 10x,\ -8y,\ 2z\rangle\end{aligned}$$

$$\begin{aligned}\nabla F(-3, 2, 4) &= \langle 10(-3),\ -8(2),\ 2(4)\rangle \\ &= \langle -30, -16, 8\rangle\end{aligned}$$

An equation for the tangent plane is given by:

$$\begin{aligned}\left(F_x\big|_P\right)(x - x_0) + \left(F_y\big|_P\right)(y - y_0) + \left(F_z\big|_P\right)(z - z_0) &= 0 \\ (-30)(x - (-3)) + (-16)(y - 2) + (8)(z - 4) &= 0\end{aligned}$$

$$\boxed{\begin{aligned}-30(x+3) - 16(y-2) + 8(z-4) &= 0 \\ \text{or}\quad 15(x+3) + 8(y-2) - 4(z-4) &= 0 \\ \text{or}\quad -30x - 16y + 8z - 90 &= 0 \\ \text{or}\quad 15x + 8y - 4z + 45 &= 0\end{aligned}}$$

9) Find all critical points of $f(x,y) = 2x^3 + 2y^3 + 3y^2 - 54x - 12y + 1$, and classify each one as a local maximum, a local minimum, or a saddle point. Show all work, as we have done in class. You do <u>not</u> have to find the corresponding function values. (27 points)

<u>Step 1</u>: Find any critical points (CPs).

$$f_x(x,y) = 6x^2 - 54 \qquad \left(\text{never DNE; is continuous on } \mathbf{R}^2\right)$$
$$f_y(x,y) = 6y^2 + 6y - 12 \qquad \left(\text{never DNE; is continuous on } \mathbf{R}^2\right)$$

Solve the system: $\begin{cases} 6x^2 - 54 = 0 \\ 6y^2 + 6y - 12 = 0 \end{cases}$

If we divide both sides of both equations by 6, we obtain the simpler system:

$$\begin{cases} x^2 - 9 = 0 \\ y^2 + y - 2 = 0 \end{cases}$$

If we solve the first equation for x, we get:

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

If we solve the second equation for y, we get:

$$\begin{aligned} y^2 + y - 2 &= 0 \\ (y+2)(y-1) &= 0 \end{aligned}$$

$$\begin{aligned} y + 2 &= 0 \\ y &= -2 \end{aligned} \quad \text{or} \quad \begin{aligned} y - 1 &= 0 \\ y &= 1 \end{aligned}$$

Each of our x values can be paired with each of our y values.

Our critical points (CPs) are:

$(-3,-2)$, $(-3,1)$, $(3,-2)$, and $(3,1)$.

(Note that they are in the domain of f.)

Step 2: Find f_{xx} and D.

Remember,

$$f_x(x,y) = 6x^2 - 54$$
$$f_y(x,y) = 6y^2 + 6y - 12$$

$$f_{xx}(x,y) = 12x$$
$$f_{xy}(x,y) = 0 \qquad \left(= f_{yx}(x,y);\ \text{note that "0" is continuous.}\right)$$
$$f_{yy}(x,y) = 12y + 6$$

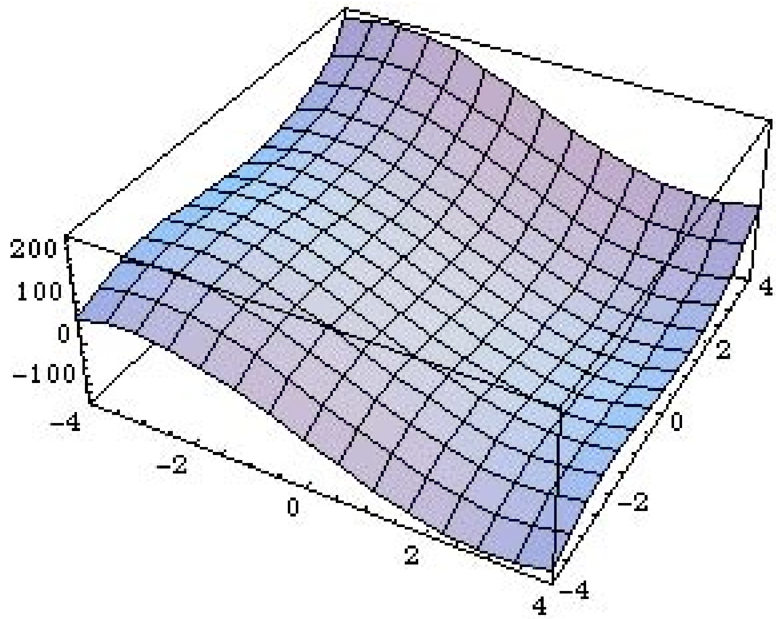
$$\begin{aligned} D &= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \\ &= \begin{vmatrix} 12x & 0 \\ 0 & 12y+6 \end{vmatrix} \\ &= (12x)(12y+6) - 0 \\ &= (12x)\left[6(2y+1)\right] \\ &= 72x(2y+1) \end{aligned}$$

Step 3: Classify the critical points (CPs).

Critical Point	$D = 72x(2y+1)$	$f_{xx} = 12x$	Classification
$(-3,-2)$	$D = (72)(-3)\left[2(-2)+1\right]$ $= (+)(-)[-]$ $= +$ (specifically, 648)	$f_{xx} = 12(-3)$ $= -36$ < 0 ("−") Think: "Concave down" $(\cap)$	**Local Maximum**
$(-3,1)$	$D = (72)(-3)\left[2(1)+1\right]$ $= (+)(-)[+]$ $= -$ (specifically, -648)	irrelevant	**Saddle Point**
$(3,-2)$	$D = (72)(3)\left[2(-2)+1\right]$ $= (+)(+)[-]$ $= -$ (specifically, -648)	irrelevant	**Saddle Point**
$(3,1)$	$D = (72)(3)\left[2(1)+1\right]$ $= (+)(+)[+]$ $= +$ (specifically, 648)	$f_{xx} = 12(3)$ $= 36$ > 0 ("+") Think: "Concave up" $(\cup)$	**Local Minimum**

Note: The corresponding points of interest on the graph are:
$(-3,-2,129)$, $(-3,1,102)$, $(3,-2,-87)$, and $(3,1,-114)$.

Here is a graph:



10) Find the absolute maximum and absolute minimum of $f(x,y,z)=3x-2y+5z$ subject to the constraint $9x^2+y^2+\frac{15}{2}z^2=1$ using the method of Lagrange multipliers. Your answers will be ordered triples in the domain of f; label which one corresponds to the absolute maximum and which one corresponds to the absolute minimum. Give exact, simplified answers, and rationalize denominators. (25 points)

Take the constraint equation, and isolate 0 on one side:

$$9x^2+y^2+\frac{15}{2}z^2=1 \quad \left(\text{Its graph is an ellipse centered at the origin.}\right)$$

$$\underbrace{9x^2+y^2+\frac{15}{2}z^2-1}_{g(x,y,z)}=0$$

Solve $\begin{cases} \nabla f(x,y,z)=\lambda\nabla g(x,y,z) \\ g(x,y,z)=0 \end{cases}$

$$\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$$
$$\left\langle f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z)\right\rangle=\lambda\left\langle g_x(x,y,z),\ g_y(x,y,z),\ g_z(x,y,z)\right\rangle$$
$$\left\langle 3,\ -2,\ 5\right\rangle=\lambda\left\langle 18x,\ 2y,\ 15z\right\rangle$$

$$\text{Solve}\begin{cases} 3=\lambda(18x) & (\text{Eq.1}) \\ -2=\lambda(2y) & (\text{Eq.2}) \\ 5=\lambda(15z) & (\text{Eq.3}) \\ 9x^2+y^2+\dfrac{15}{2}z^2-1=0 & (\text{Eq.C}) \end{cases}$$

Solve the first three equations for λ. Observe that $x=0$, $y=0$, and $z=0$ cannot solve the first three equations, so we may assume that x, y, and z are nonzero.

$$(\text{Eq.1}) \;\Rightarrow\; \lambda=\frac{3}{18x} \;\Rightarrow\; \lambda=\frac{1}{6x}$$
$$(\text{Eq.2}) \;\Rightarrow\; \lambda=\frac{-2}{2y} \;\Rightarrow\; \lambda=-\frac{1}{y}$$
$$(\text{Eq.3}) \;\Rightarrow\; \lambda=\frac{5}{15z} \;\Rightarrow\; \lambda=\frac{1}{3z}$$

Equate our expressions for λ.

$$\lambda=\frac{1}{6x}=-\frac{1}{y}=\frac{1}{3z}$$

Equate the reciprocals of the last three expressions:

$$6x=-y=3z$$

We may then solve for y and z in terms of x:

$$6x=-y \;\Rightarrow\; y=-6x$$

$$6x=3z \;\Rightarrow\; z=2x$$

Substitute $y = -6x$ and $z = 2x$ into (Eq.C).

$$\begin{aligned} 9x^2 + y^2 + \frac{15}{2}z^2 - 1 &= 0 \\ 9x^2 + (-6x)^2 + \frac{15}{2}(2x)^2 - 1 &= 0 \\ 9x^2 + 36x^2 + \frac{15}{2}(4x^2) - 1 &= 0 \\ 9x^2 + 36x^2 + 30x^2 - 1 &= 0 \\ 75x^2 &= 1 \\ x^2 &= \frac{1}{75} \\ x &= \pm\sqrt{\frac{1}{75}} \\ x &= \pm\frac{1}{5\sqrt{3}} \quad \text{or} \quad \pm\frac{\sqrt{3}}{15} \end{aligned}$$

Find our candidate points. Remember: $y = -6x$ and $z = 2x$.

$$x = \frac{\sqrt{3}}{15} \quad \Rightarrow \quad y = -6\left(\frac{\sqrt{3}}{15}\right) = -\frac{2\sqrt{3}}{5} \quad \text{and} \quad z = 2\left(\frac{\sqrt{3}}{15}\right) = \frac{2\sqrt{3}}{15}.$$

$$x = -\frac{\sqrt{3}}{15} \quad \Rightarrow \quad y = -6\left(-\frac{\sqrt{3}}{15}\right) = \frac{2\sqrt{3}}{5} \quad \text{and} \quad z = 2\left(-\frac{\sqrt{3}}{15}\right) = -\frac{2\sqrt{3}}{15}.$$

Candidate point 1: $\left(\frac{\sqrt{3}}{15}, -\frac{2\sqrt{3}}{5}, \frac{2\sqrt{3}}{15}\right)$

Function value from $f(x,y,z) = 3x - 2y + 5z$:

$$\begin{aligned} 3\left(\frac{\sqrt{3}}{15}\right) - 2\left(-\frac{2\sqrt{3}}{5}\right) + 5\left(\frac{2\sqrt{3}}{15}\right) &= \frac{3\sqrt{3}}{15} + \frac{4\sqrt{3}}{5} + \frac{10\sqrt{3}}{15} \\ &= \frac{3\sqrt{3}}{15} + \frac{12\sqrt{3}}{15} + \frac{10\sqrt{3}}{15} \\ &= \frac{25\sqrt{3}}{15} \\ &= \frac{5\sqrt{3}}{3} \end{aligned}$$

Candidate point 2:

$$\left(-\frac{\sqrt{3}}{15}, \frac{2\sqrt{3}}{5}, -\frac{2\sqrt{3}}{15}\right)$$

Function value from $f(x,y,z) = 3x - 2y + 5z$:

$$3\left(-\frac{\sqrt{3}}{15}\right) - 2\left(\frac{2\sqrt{3}}{5}\right) + 5\left(-\frac{2\sqrt{3}}{15}\right) = -\frac{3\sqrt{3}}{15} - \frac{4\sqrt{3}}{5} - \frac{10\sqrt{3}}{15}$$
$$= -\frac{5\sqrt{3}}{3}$$

Our candidates are both in the domain of f.

We know that the candidates for local extrema we find are absolute extrema, because there are only two, the graph of the constraint is a closed surface, and f is continuous on it. We may identify which is the absolute maximum and which is the absolute minimum by evaluating and comparing values of f at these candidate points, or (in this case) by applying a sign analysis: Observe that evaluating $3x - 2y + 5z$ at the first candidate point yields three positive terms, while evaluating it at the second candidate point yields three negative terms.

$\left(\frac{\sqrt{3}}{15}, -\frac{2\sqrt{3}}{5}, \frac{2\sqrt{3}}{15}\right)$ corresponds to the absolute maximum, and
$\left(-\frac{\sqrt{3}}{15}, \frac{2\sqrt{3}}{5}, -\frac{2\sqrt{3}}{15}\right)$ corresponds to the absolute minimum.