# QUIZ 3 (SECTIONS 16.3-16.9) SOLUTIONS

**MATH 252 – FALL 2008 – KUNIYUKI**

**SCORED OUT OF 125 POINTS ⇒ MULTIPLIED BY 0.84 ⇒ 105% POSSIBLE**

1) Let $f(r,t) = t^2 \sin\left(\frac{r}{t}\right)$. Find $f_r(r,t)$. (5 points)

$$
\begin{aligned}
f_r(r,t) &= D_r\left[\underbrace{t^2}_{\text{"\#"}} \sin\left(\frac{r}{t}\right)\right] \\
&= t^2 \cdot D_r\left[\sin\left(\frac{r}{t}\right)\right] \\
&= t^2 \cdot \left[\cos\left(\frac{r}{t}\right)\right] \cdot D_r\left(\frac{r}{t}\right) \\
&= t^2 \cdot \left[\cos\left(\frac{r}{t}\right)\right] \cdot D_r\left(\underbrace{\frac{1}{t}}_{\text{"\#"}} \cdot r\right) \\
&= t^2 \cdot \left[\cos\left(\frac{r}{t}\right)\right] \cdot \frac{1}{t} \\
&= t\cos\left(\frac{r}{t}\right)
\end{aligned}
$$

2) Let $f(x,y,z)=e^{xyz}$. Find $f_y(x,y,z)$ and use that to find $f_{yz}(x,y,z)$. (8 points)

$$\begin{aligned} f_y(x,y,z) &= D_y\left[e^{xyz}\right] \\ &= \left[e^{xyz}\right]\cdot\left[D_y(xyz)\right] \\ &= \left[e^{xyz}\right]\cdot\left[D_y\left(\underbrace{xz}_{\text{"\#"}}\cdot y\right)\right] \\ &= e^{xyz}\cdot xz \\ &= xze^{xyz} \end{aligned}$$

$$\begin{aligned} f_{yz}(x,y,z) &= D_z\left[f_y(x,y,z)\right] \\ &= D_z\left[\underbrace{x}_{\text{"\#"}}\, ze^{xyz}\right] \\ &= x\cdot D_z\left[ze^{xyz}\right] \end{aligned}$$

We will use a Product Rule for Differentiation.

$$\begin{aligned} &= x\left(\underbrace{\left[D_z(z)\right]}_{=(1)}\cdot\left[e^{xyz}\right] + \left[z\right]\cdot\left[D_z\left(e^{xyz}\right)\right]\right) \\ &= x\left[e^{xyz}+z\cdot D_z\left(e^{xyz}\right)\right] \\ &= x\left[e^{xyz}+z\cdot e^{xyz}\cdot D_z\left(\underbrace{xy}_{\text{"\#"}}\, z\right)\right] \\ &= x\left[e^{xyz}+z\cdot e^{xyz}\cdot xy\right] \\ &= x\left(e^{xyz}+xyze^{xyz}\right) \\ &= \boxed{xe^{xyz}(1+xyz),\ \text{or}\ \ xe^{xyz}+x^2yze^{xyz}} \end{aligned}$$

3) Assume that $f$ is a function of $s$ and $t$. Write the limit definition of $f_s(s,t)$ using the notation from class. (4 points)

$$\boxed{f_s(s,t)=\lim_{h\to 0}\frac{f(s+h,t)-f(s,t)}{h}}$$

4) Find $\dfrac{\partial z}{\partial y}$ if $z = f(x,y)$ is a differentiable function described implicitly by the equation $\tan(y^3 z) = x^2 - yz$. Use the Calculus III formula given in class. Simplify. (12 points)

First, isolate 0 on one side: $\underbrace{\tan(y^3 z) - x^2 + yz}_{\text{Let this be } F(x,y,z)} = 0$

Find $\dfrac{\partial z}{\partial y}$. When using the formula, treat *x*, *y*, and *z* as independent variables.

$$\frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)}$$

$$= -\frac{D_y\left[\tan(y^3 z) - \overbrace{x^2}^{\text{"\#"}} + y\overbrace{z}^{\text{"\#"}}\right]}{D_z\left[\tan(y^3 z) - \underbrace{x^2}_{\text{"\#"}} + \underbrace{y}_{\text{"\#"}}z\right]}$$

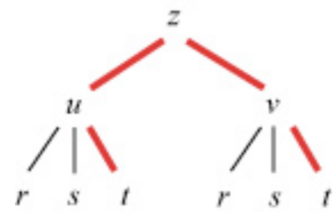
$$= -\frac{\left[\sec^2(y^3 z)\right]\cdot\left[D_y\left(y^3\overbrace{z}^{\text{"\#"}}\right)\right] + z}{\left[\sec^2(y^3 z)\right]\cdot\left[D_z\left(\underbrace{y^3}_{\text{"\#"}}z\right)\right] + y}$$

$$= -\frac{\left[\sec^2(y^3 z)\right]\cdot\left[z\cdot D_y(y^3)\right] + z}{\left[\sec^2(y^3 z)\right]\cdot\left[y^3\right] + y}$$

$$= -\frac{\left[\sec^2(y^3 z)\right]\cdot\left[z\cdot 3y^2\right] + z}{\left[\sec^2(y^3 z)\right]\cdot\left[y^3\right] + y}$$

$$= \boxed{-\frac{3y^2 z\sec^2(y^3 z) + z}{y^3\sec^2(y^3 z) + y} \quad \text{or} \quad -\frac{z\left[3y^2\sec^2(y^3 z) + 1\right]}{y\left[y^2\sec^2(y^3 z) + 1\right]}}$$

5) Let $f$, $g$, and $h$ be differentiable functions such that $z = f(u, v)$, $u = g(r, s, t)$, and $v = h(r, s, t)$. Use the Chain Rule to write an expression for $\frac{\partial z}{\partial t}$. (5 points)




$$\boxed{\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t}}$$

6) The temperature at any point $(x, y)$ in the $xy$-plane is given by $f(x, y) = 5x^2 y + y^3$ in degrees Fahrenheit. Assume $x$ and $y$ are measured in meters. Give appropriate units in your answers. (30 points total)

a) Find the **maximum** rate of change of temperature at the point $(2, -3)$. Approximate your final answer to four significant digits. (10 points)

$$\begin{aligned}\nabla f(x, y) &= \langle f_x(x, y),\ f_y(x, y)\rangle \\ &= \langle D_x(5x^2 y + y^3),\ \ D_y(5x^2 y + y^3)\rangle \\ &= \langle 10xy,\ 5x^2 + 3y^2\rangle\end{aligned}$$

$$\begin{aligned}\nabla f(2, -3) &= \langle 10(2)(-3),\ 5(2)^2 + 3(-3)^2\rangle \\ &= \langle -60, 47\rangle\end{aligned}$$

The length of the gradient of $f$ at $(2, -3)$ gives the **maximum** rate of change of temperature at that point.

$$\begin{aligned}\|\nabla f(2, -3)\| &= \|\langle -60, 47\rangle\| \\ &= \sqrt{(-60)^2 + (47)^2} \\ &= \sqrt{5809} \\ &\approx 76.22\end{aligned}$$

Answer: $\boxed{\text{About } 76.22\ \frac{^\circ\text{F}}{\text{m}}}$

b) Find the rate of change of temperature at $(2,-3)$ in the direction of $4\mathbf{i}+\mathbf{j}$. Approximate your final answer to four significant digits. (10 points)

Let **a** be the given direction vector $4\mathbf{i}+\mathbf{j}$, or $\langle 4,1\rangle$.

Find the unit vector **u** in the direction of **a**.

$$\begin{aligned}\mathbf{u} &= \frac{\mathbf{a}}{\|\mathbf{a}\|}\\ &= \frac{\langle 4,1\rangle}{\|\langle 4,1\rangle\|}\\ &= \frac{\langle 4,1\rangle}{\sqrt{(4)^2+(1)^2}}\\ &= \frac{\langle 4,1\rangle}{\sqrt{17}} \quad \text{or} \quad \frac{1}{\sqrt{17}}\langle 4,1\rangle \quad \text{or} \quad \frac{\sqrt{17}}{17}\langle 4,1\rangle\end{aligned}$$

The directional derivative at $(2,-3)$ in the direction of **u** is:

$$\begin{aligned}D_{\mathbf{u}} f(2,-3) &= \nabla f(2,-3)\bullet\mathbf{u}\\ &= \langle -60, 47\rangle \bullet \left(\frac{1}{\sqrt{17}}\langle 4,1\rangle\right)\\ &= \frac{1}{\sqrt{17}}\left(\langle -60,47\rangle\bullet\langle 4,1\rangle\right)\\ &= \frac{1}{\sqrt{17}}\left[(-60)(4)+(47)(1)\right]\\ &= \frac{1}{\sqrt{17}}\left[-193\right]\\ &= -\frac{193}{\sqrt{17}} \quad \text{or} \quad -\frac{193\sqrt{17}}{17}\\ &\approx -46.81\end{aligned}$$

Answer: $\boxed{\text{About } -46.81\,\frac{^{\circ}\text{F}}{\text{m}}}$

c) Use differentials to linearly approximate the change in temperature if $(x, y)$ changes from $(2, -3)$ to $(2.03, -3.01)$. (7 points)

From part a), we know that $\nabla f(2,-3) = \langle -60, 47 \rangle$. In particular,

$$f_x(2,-3) = -60 \text{, and}$$
$$f_y(2,-3) = 47.$$

Find the changes in *x* and *y*:

$$\begin{aligned} dx &= \text{new } x - \text{old } x \\ &= 2.03 - 2 \\ &= 0.03 \end{aligned}$$

$$\begin{aligned} dy &= \text{new } y - \text{old } y \\ &= -3.01 - (-3) \\ &= -0.01 \end{aligned}$$

The approximate change in $f$ is given by:

$$\begin{aligned} df &= \left[ f_x(2,-3) \right] dx + \left[ f_y(2,-3) \right] dy \\ &= [-60][0.03] + [47][-0.01] \\ &= -2.27 \end{aligned}$$

Note: Actual change $= f(2.03, -3.01) - f(2,-3) \approx -2.29045$.

Answer: $-2.27\ ^{\circ}\text{F}$

d) In what direction does the temperature **decrease** most rapidly at $(2, -3)$? Give an appropriate non-**0** direction vector. (3 points)

We want a vector that points in the direction opposite from the gradient vector, $\nabla f(2,-3) = \langle -60, 47 \rangle$.

Answer: $\langle 60\ [\text{m}], -47\ [\text{m}] \rangle$, or any positive scalar multiple of this.

7) Find an equation for the tangent plane to the graph of the equation $z^3 = x - xy$ at the point $P(8, -26, 6)$. (10 points)

(You may check that the coordinates of $P$ satisfy the given equation, meaning that $P$ lies on the graph of the equation.)

Isolate 0 on one side of the given equation.

$$\underbrace{z^3 - x + xy}_{=F(x,y,z)} = 0$$

A normal vector for the desired tangent plane is given by $\nabla F(8, -26, 6)$.

$$\begin{aligned}\nabla F\big|_P &= \left\langle F_x\big|_P,\ F_y\big|_P,\ F_z\big|_P \right\rangle \\ &= \left\langle -1 + y,\ x,\ 3z^2 \right\rangle \text{ or } \left\langle y - 1,\ x,\ 3z^2 \right\rangle\end{aligned}$$

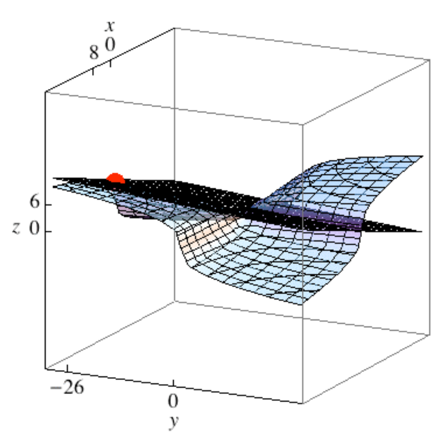
$$\begin{aligned}\nabla F(8, -26, 6) &= \left\langle -1 + (-26),\ 8,\ 3(6)^2 \right\rangle \\ &= \left\langle -27, 8, 108 \right\rangle\end{aligned}$$

An equation for the tangent plane is given by:

$$\begin{aligned}\left(F_x\big|_P\right)(x - x_0) + \left(F_y\big|_P\right)(y - y_0) + \left(F_z\big|_P\right)(z - z_0) &= 0 \\ (-27)(x - 8) + (8)(y - (-26)) + (108)(z - 6) &= 0\end{aligned}$$

$$\boxed{\begin{aligned} -27(x-8) + 8(y+26) + 108(z-6) &= 0 \\ \text{or}\quad 27(x-8) - 8(y+26) - 108(z-6) &= 0 \\ \text{or}\quad -27x + 8y + 108z - 224 &= 0 \\ \text{or}\quad 27x - 8y - 108z + 224 &= 0 \end{aligned}}$$

Here's a graph:



8) Find all critical points of $f(x,y) = 2x^4 + y^2 - 12xy$, and classify each one as a local maximum, a local minimum, or a saddle point. Show all work, as we have done in class. You do <u>not</u> have to find the corresponding function values. (26 points)

<u>Step 1</u>: Find any critical points (CPs).

$$f_x(x,y) = 8x^3 - 12y \quad \left(\text{never DNE; is continuous on } \mathbf{R}^2\right)$$
$$f_y(x,y) = 2y - 12x \quad \left(\text{never DNE; is continuous on } \mathbf{R}^2\right)$$

Solve the system: $\begin{cases} 8x^3 - 12y = 0 \\ 2y - 12x = 0 \end{cases}$

(Optional.) If we divide both sides of the first equation by 4 and divide both sides of the second equation by 2, we obtain the simpler system:

$$\begin{cases} 2x^3 - 3y = 0 \\ y - 6x = 0 \end{cases}$$

We can solve this using the Substitution Method.

If we solve the second equation for $y$, we get:

$$y = 6x$$

Incorporate this into the first equation.

$$2x^3 - 3y = 0, \quad y = 6x \quad \Rightarrow$$
$$2x^3 - 3(6x) = 0$$
$$2x^3 - 18x = 0$$
$$2x\left(x^2 - 9\right) = 0$$
$$2x(x+3)(x-3) = 0$$

$x = 0$ or $x + 3 = 0$, $x = -3$ or $x - 3 = 0$, $x = 3$

Find the corresponding $x$-values.

Using $y = 6x$,

$x = 0 \Rightarrow y = 0 \Rightarrow (0, 0)$

$x = -3 \Rightarrow y = -18 \Rightarrow (-3, -18)$

$x = 3 \Rightarrow y = 18 \Rightarrow (3, 18)$

Our critical points (CPs) are:

$$(0,0),\ (-3,-18),\text{ and }(3,18).$$

(Note that they are in the domain of $f$.)

Step 2: Find $f_{xx}$ and $D$.

Remember,

$$f_x(x,y) = 8x^3 - 12y$$
$$f_y(x,y) = 2y - 12x$$

$$f_{xx}(x,y) = 24x^2$$
$$f_{xy}(x,y) = -12 \qquad \left(= f_{yx}(x,y);\ \text{note that "}-12\text{" is continuous.}\right)$$
$$f_{yy}(x,y) = 2$$

$$\begin{aligned} D &= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \\ &= \begin{vmatrix} 24x^2 & -12 \\ -12 & 2 \end{vmatrix} \\ &= (24x^2)(2) - (-12)(-12) \\ &= 48x^2 - 144 \end{aligned}$$

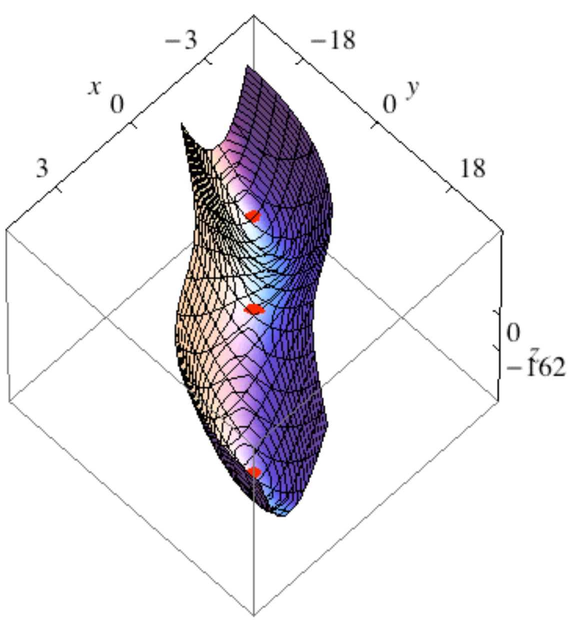
<u>Step 3</u>: Classify the critical points (CPs).

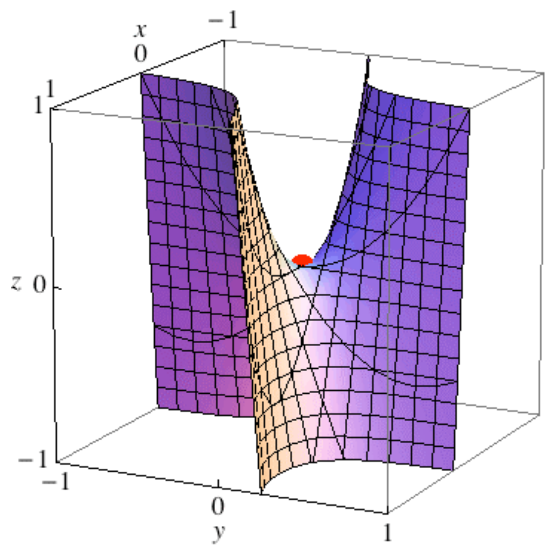
| Critical Point | $D = 48x^2 - 144$ | $f_{xx} = 24x^2$ | Classification |
|---|---|---|---|
| $(0, 0)$ | $D = 48(0)^2 - 144$ <br> $= -144$ <br> $< 0 \quad ("-")$ | irrelevant | **Saddle Point** |
| $(-3, -18)$ | $D = 48(-3)^2 - 144$ <br> $= 288$ <br> $> 0 \quad ("+")$ | $f_{xx} = 24(-3)^2$ <br> $= 216$ <br> $> 0 \quad ("+")$ <br> Think: "Concave up" $(\cup)$ | **Local Minimum** |
| $(3, 18)$ | $D = 48(3)^2 - 144$ <br> $= 288$ <br> $> 0 \quad ("+")$ | $f_{xx} = 24(3)^2$ <br> $= 216$ <br> $> 0 \quad ("+")$ <br> Think: "Concave up" $(\cup)$ | **Local Minimum** |

Note 1: The corresponding points of interest on the graph are:
$(0, 0, 0)$, $(-3, -18, -162)$, and $(3, 18, -162)$.

Note 2: Instead of using $f_{xx} = 24x^2$, it may be easier to use $f_{yy}(x, y) = 2$, which is positive everywhere in the *xy*-plane and in particular at $(-3, -18)$ and $(3, 18)$.

Here are graphs. The second graph zooms in on the saddle point at the origin.



9) The function $f$, where $f(x,y,z)=(x-3)^2+(y-5)^2+(z-7)^2$, gives the squared distance of the point $(x,y,z)$ from the point $(3,5,7)$ in *xyz*-space. Using the method of Lagrange multipliers, find the point on the unit sphere $x^2+y^2+z^2=1$ that is closest to the point $(3,5,7)$. The point will lie in Octant I; you may use this fact without proof. You do <u>not</u> have to prove that your candidate point corresponds to an absolute minimum of $f$ under the constraint. Give an exact, simplified answer, and rationalize denominators. (25 points)

Take the constraint equation, and isolate 0 on one side:

$$x^2+y^2+z^2=1$$
$$\underbrace{x^2+y^2+z^2-1}_{g(x,y,z)}=0$$

Solve $\begin{cases}\nabla f(x,y,z)=\lambda\nabla g(x,y,z)\\ g(x,y,z)=0\end{cases}$

$$\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$$
$$\left\langle f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z)\right\rangle=\lambda\left\langle g_x(x,y,z),\ g_y(x,y,z),\ g_z(x,y,z)\right\rangle$$
$$\left\langle 2(x-3),\ 2(y-5),\ 2(z-7)\right\rangle=\lambda\left\langle 2x,\ 2y,\ 2z\right\rangle$$

Solve $\begin{cases}2(x-3)=\lambda(2x) & \text{(Eq.1)}\\ 2(y-5)=\lambda(2y) & \text{(Eq.2)}\\ 2(z-7)=\lambda(2z) & \text{(Eq.3)}\\ x^2+y^2+z^2-1=0 & \text{(Eq.C)}\end{cases}$

Solve the first three equations for $\lambda$. Observe that $x=0$, $y=0$, and $z=0$ cannot solve the first three equations, so we may assume that *x*, *y*, and *z* are nonzero.

$$\text{(Eq.1)}\ \Rightarrow\ \lambda=\frac{2(x-3)}{2x}\ \Rightarrow\ \lambda=\frac{x-3}{x}$$
$$\text{(Eq.2)}\ \Rightarrow\ \lambda=\frac{2(y-5)}{2y}\ \Rightarrow\ \lambda=\frac{y-5}{y}$$
$$\text{(Eq.3)}\ \Rightarrow\ \lambda=\frac{2(z-7)}{2z}\ \Rightarrow\ \lambda=\frac{z-7}{z}$$

Equate our expressions for $\lambda$.

$$\lambda = \frac{x-3}{x} = \frac{y-5}{y} = \frac{z-7}{z}$$

We may then solve for *y* and *z* in terms of *x*.
Remember that we may assume that *x*, *y*, and *z* are nonzero.

$$\frac{x-3}{x} = \frac{y-5}{y}$$
$$\cancel{x}\, y\left(\frac{x-3}{\cancel{x}}\right) = x\,\cancel{y}\left(\frac{y-5}{\cancel{y}}\right)$$
$$y(x-3) = x(y-5)$$
$$\cancel{xy} - 3y = \cancel{xy} - 5x$$
$$-3y = -5x$$
$$y = \frac{5}{3}x$$

$$\frac{x-3}{x} = \frac{z-7}{z}$$
$$\cancel{x}\, z\left(\frac{x-3}{\cancel{x}}\right) = x\,\cancel{z}\left(\frac{z-7}{\cancel{z}}\right)$$
$$z(x-3) = x(z-7)$$
$$\cancel{xz} - 3z = \cancel{xz} - 7x$$
$$-3z = -7x$$
$$z = \frac{7}{3}x$$

Substitute $y = \frac{5}{3}x$ and $z = \frac{7}{3}x$ into (Eq.C).

$$x^2 + y^2 + z^2 - 1 = 0, \quad y = \frac{5}{3}x \text{ and } z = \frac{7}{3}x \quad \Rightarrow$$
$$x^2 + \left(\frac{5}{3}x\right)^2 + \left(\frac{7}{3}x\right)^2 - 1 = 0$$
$$x^2 + \frac{25}{9}x^2 + \frac{49}{9}x^2 - 1 = 0$$
$$\frac{9}{9}x^2 + \frac{25}{9}x^2 + \frac{49}{9}x^2 = 1$$
$$\frac{83}{9}x^2 = 1$$
$$x^2 = \frac{9}{83}$$
$$x = \pm\sqrt{\frac{9}{83}}$$
$$x = \pm\frac{3}{\sqrt{83}}$$
$$x = \pm\frac{3\sqrt{83}}{83}$$

Take $x = \frac{3\sqrt{83}}{83}$. (We restrict our attention to Octant I.)

Find our candidate point. Remember: $y = \frac{5}{3}x$ and $z = \frac{7}{3}x$.

$$x = \frac{3\sqrt{83}}{83} \quad \Rightarrow \quad y = \frac{5}{\cancel{3}}\left(\frac{\cancel{3}\sqrt{83}}{83}\right) = \frac{5\sqrt{83}}{83} \quad \text{and} \quad z = \frac{7}{\cancel{3}}\left(\frac{\cancel{3}\sqrt{83}}{83}\right) = \frac{7\sqrt{83}}{83}.$$

Candidate point: $\left(\frac{3\sqrt{83}}{83}, \frac{5\sqrt{83}}{83}, \frac{7\sqrt{83}}{83}\right)$.

Our candidate is in the domain of $f$, which is $\mathbf{R}^3$.

If we had not restricted our attention to Octant I, we would have obtained two candidate points: $\left(\frac{3\sqrt{83}}{83}, \frac{5\sqrt{83}}{83}, \frac{7\sqrt{83}}{83}\right)$ and $\left(-\frac{3\sqrt{83}}{83}, -\frac{5\sqrt{83}}{83}, -\frac{7\sqrt{83}}{83}\right)$.
We know that our candidates for local extrema are absolute extrema, because there are only two, the graph of the constraint is a closed surface (in this case, a sphere), and $f$ is continuous on it. We may identify which is the absolute maximum and which is the absolute minimum by evaluating and comparing values of $f$ at these candidate points. It turns out that:

$$f\left(\frac{3\sqrt{83}}{83}, \frac{5\sqrt{83}}{83}, \frac{7\sqrt{83}}{83}\right) = 84 - 2\sqrt{83} \approx 65.8, \text{ while}$$

$$f\left(-\frac{3\sqrt{83}}{83}, -\frac{5\sqrt{83}}{83}, -\frac{7\sqrt{83}}{83}\right) = 84 + 2\sqrt{83} \approx 102.2.$$

Therefore, $\left(\frac{3\sqrt{83}}{83}, \frac{5\sqrt{83}}{83}, \frac{7\sqrt{83}}{83}\right)$ is the absolute minimum of $f$ on the given unit sphere. In other words, it is the point on the unit sphere that is closest to the point $(3, 5, 7)$; the point $\left(\frac{3\sqrt{83}}{83}, \frac{5\sqrt{83}}{83}, \frac{7\sqrt{83}}{83}\right)$ is the point on the sphere that has the absolute minimum squared distance (and therefore the absolute minimum distance) from the point $(3, 5, 7)$.

Answer: $\boxed{\left(\frac{3\sqrt{83}}{83}, \frac{5\sqrt{83}}{83}, \frac{7\sqrt{83}}{83}\right)}$

<u>Note</u>: Observe that the origin, this point on the sphere, and the point $(3, 5, 7)$ all lie on a straight line. Our visual intuition leads us to believe that we have, in fact, found the correct point.