QUIZ 4 (CHAPTER 17) SOLUTIONS

MATH 252 – FALL 2007 – KUNIYUKI
SCORED OUT OF 125 POINTS ⇒ MULTIPLIED BY 0.84 ⇒ 105% POSSIBLE

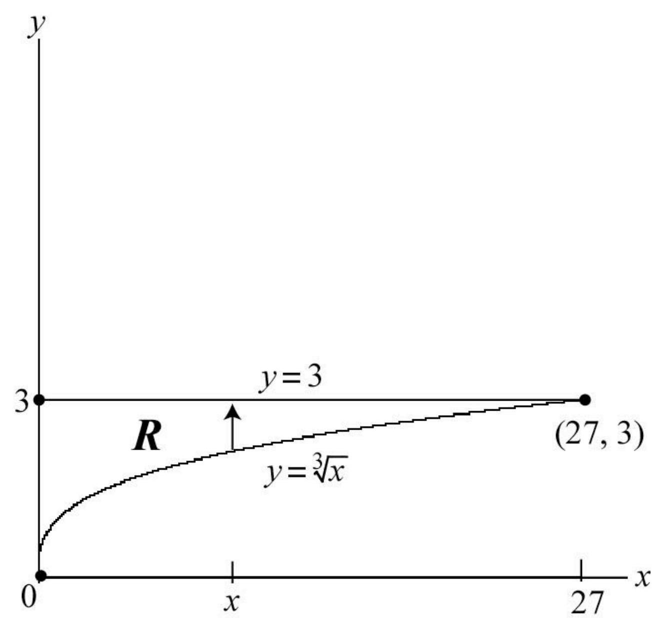
1) Reverse the order of integration, and **evaluate** the resulting double integral: $\int_0^{27}\int_{\sqrt[3]{x}}^{3} \cos\left(1+y^4\right)dy\,dx$. Give a simplified exact answer; do not approximate. Sketch the region of integration. (20 points)

Sketch the region of integration, R.

It may help to rewrite the double integral with labels:

$$\int_{x=0}^{x=27}\int_{y=\sqrt[3]{x}}^{y=3} \cos\left(1+y^4\right)dy\,dx$$

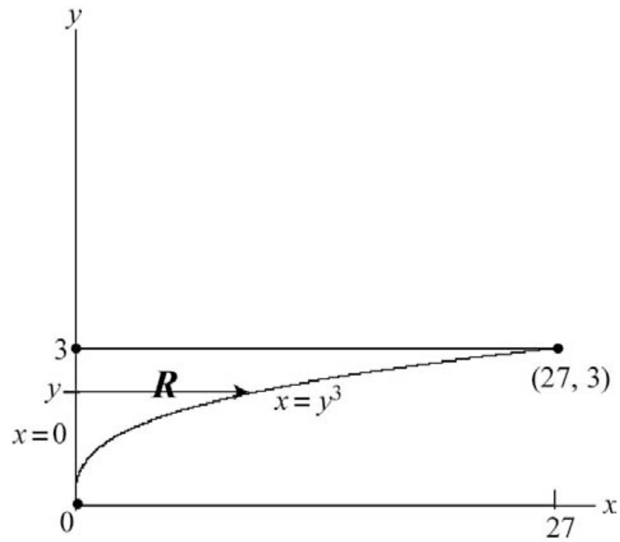
We aim / fix x, shoot y, and slide x.



Reverse the order of integration.

Solve $y=\sqrt[3]{x}$ for x:

$x = y^3$



We aim y, shoot x, and slide y.

$$\int_{y=0}^{y=3} \int_{x=0}^{x=y^3} \cos\left(1+y^4\right) dx\, dy$$

$$= \int_{y=0}^{y=3} \left[x\cos\left(1+y^4\right) \right]_{x=0}^{x=y^3} dy$$

$$= \int_0^3 \left(\left[y^3 \cos\left(1+y^4\right) \right] - \left[0 \right] \right) dy$$

$$= \int_0^3 y^3 \cos\left(1+y^4\right) dy$$

We now perform a standard u-substitution.

Let $u = 1 + y^4$

$$du = 4y^3\, dy \quad \Rightarrow \quad y^3\, dy = \frac{1}{4} du$$

You can also "Compensate":

$$\int_0^3 y^3 \cos\left(1+y^4\right) dy = \frac{1}{4} \int_0^3 4y^3 \cos\left(1+y^4\right) dy$$

Change the limits of integration:

$$y = 0 \quad \Rightarrow \quad u = 1 + (0)^4 \quad \Rightarrow \quad u = 1$$

$$y = 3 \quad \Rightarrow \quad u = 1 + (3)^4 \quad \Rightarrow \quad u = 82$$

$$= \int_{u=1}^{u=82} \cos u \cdot \frac{1}{4}\, du$$

$$= \frac{1}{4} \left[\sin u \right]_1^{82}$$

$$= \boxed{\frac{1}{4}\left(\sin 82 - \sin 1\right)} \qquad \left(\text{Note: This is about } -0.13206.\right)$$

2) Let R be the region in the xy-plane that is bounded by the rectangle with vertices $(1,3)$, $(7,3)$, $(7,5)$, and $(1,5)$. **Set up a double integral** for the surface area of the portion of the graph of $9x^2+4y^2+z^2=1000$ $(z>0)$ that lies over R. Make sure your double integral is as detailed as possible; do not leave in generic notation like R or f. Also, do not leave dA in your final answer; break it down into $d(\text{variable})\,d(\text{variable})$. Do **not** evaluate. (14 points)

Observe that the graph of $9x^2+4y^2+z^2=1000$ is an ellipsoid. With the restriction $(z>0)$, we only consider [part of] the upper half.

Solve $9x^2+4y^2+z^2=1000$ for z:

$$\begin{aligned} 9x^2+4y^2+z^2&=1000 && (z>0)\\ z^2&=1000-9x^2-4y^2 && (z>0)\\ z&=\sqrt{1000-9x^2-4y^2} \end{aligned}$$

Let $f(x,y)=\sqrt{1000-9x^2-4y^2}$. Its domain contains R.

$$S=\iint_R \sqrt{1+\left[f_x(x,y)\right]^2+\left[f_y(x,y)\right]^2}\,dA$$

$$\begin{aligned} f(x,y)&=\left(1000-9x^2-4y^2\right)^{\frac{1}{2}}\\ f_x(x,y)&=\frac{1}{2}\left(1000-9x^2-4y^2\right)^{-\frac{1}{2}}(-18x)\\ &=-\frac{9x}{\sqrt{1000-9x^2-4y^2}}\\ f_y(x,y)&=\frac{1}{2}\left(1000-9x^2-4y^2\right)^{-\frac{1}{2}}(-8y)\\ &=-\frac{4y}{\sqrt{1000-9x^2-4y^2}} \end{aligned}$$

$$S=\int_1^7\int_3^5\sqrt{1+\left[-\frac{9x}{\sqrt{1000-9x^2-4y^2}}\right]^2+\left[-\frac{4y}{\sqrt{1000-9x^2-4y^2}}\right]^2}\,dy\,dx$$

$$\text{or } S=\int_1^7\int_3^5\sqrt{1+\frac{81x^2}{1000-9x^2-4y^2}+\frac{16y^2}{1000-9x^2-4y^2}}\,dy\,dx$$

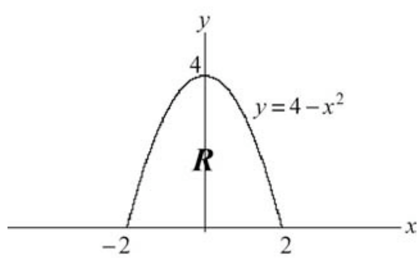
$$\text{or } S=\int_1^7\int_3^5\sqrt{1+\frac{81x^2+16y^2}{1000-9x^2-4y^2}}\,dy\,dx$$

3) Find the **center of mass** of the lamina that makes up the region in the xy-plane bounded by the x-axis and the graph of $y = 4 - x^2$. The area mass density is given by $\delta(x, y) = 3y$. Sketch the region. Give exact coordinates; do not approximate. Hint: You may use the following rule based on the Binomial Theorem: $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$. (32 points)

Let R be the region.

The x-intercepts of the graph of $y = 4 - x^2$ are given by:

$$\begin{aligned} 0 &= 4 - x^2 \\ x^2 &= 4 \\ x &= \pm 2 \quad \left(\text{i.e., the points } (-2,0) \text{ and } (2,0)\right) \end{aligned}$$



Let $\left(\bar{x}, \bar{y}\right)$ be the center of mass. By symmetry of R about the line $x = 0$ (i.e., the y-axis) and the fact that $\delta(x, y) = 3y$ is even in x, we have $\bar{x} = 0$.

Find the mass of the lamina (corresponding to region "R"):

$$\begin{aligned} m &= \iint_R \delta(x, y)\, dA \\ &= \int_{x=-2}^{x=2} \int_{y=0}^{y=4-x^2} \overbrace{3y}^{\delta} \cdot \overbrace{dy\, dx}^{dA} \\ &= 2 \int_{x=0}^{x=2} \int_{y=0}^{y=4-x^2} 3y\, dy\, dx \\ &\qquad \text{by the aforementioned symmetries} \\ &= 6 \int_{x=0}^{x=2} \int_{y=0}^{y=4-x^2} y\, dy\, dx \\ &= 6 \int_{x=0}^{x=2} \left[\frac{y^2}{2} \right]_{y=0}^{y=4-x^2} dx \end{aligned}$$

$$= 6\int_{x=0}^{x=2}\left(\left[\frac{\left(4-x^2\right)^2}{2}\right]-\left[0\right]\right)dx$$

$$= 3\int_0^2 \left(4-x^2\right)^2 dx$$

$$= 3\int_0^2 \left(16-8x^2+x^4\right)dx$$

$$= 3\left[16x-\frac{8x^3}{3}+\frac{x^5}{5}\right]_0^2$$

$$= 3\left(\left[16(2)-\frac{8(2)^3}{3}+\frac{(2)^5}{5}\right]-\left[0\right]\right)$$

$$= \frac{256}{5} \text{ or } 51.2 \left[\text{mass units}\right]$$

Find $\bar{y}$.

$\bar{y} = \dfrac{M_x}{m}$, where:

$$M_x = \iint_R y\,\delta(x,y)\,dA$$

$$= \int_{x=-2}^{x=2}\int_{y=0}^{y=4-x^2} y\cdot\overbrace{3y}^{\delta}\cdot\overbrace{dy\,dx}^{dA}$$

$$= 2\int_{x=0}^{x=2}\int_{y=0}^{y=4-x^2} 3y^2\cdot dy\,dx$$

by the aforementioned symmetries

$$= 6\int_{x=0}^{x=2}\int_{y=0}^{y=4-x^2} y^2\,dy\,dx$$

$$= 6\int_{x=0}^{x=2}\left[\frac{y^3}{3}\right]_{y=0}^{y=4-x^2} dx$$

$$= 6\int_{x=0}^{x=2}\left(\left[\frac{\left(4-x^2\right)^3}{3}\right]-\left[0\right]\right)dx$$

$$= 2\int_0^2 \left(4-x^2\right)^3 dx$$

$$= 2\int_0^2 \left[a^3-3a^2b+3ab^2-b^3\right]dx$$

$\left(\text{where } a=(4) \text{ and } b=\left(x^2\right)\right)$

$$= 2\int_0^2 \left[(4)^3-3(4)^2\left(x^2\right)+3(4)\left(x^2\right)^2-\left(x^2\right)^3\right]dx$$

(by the Hint)

$$
\begin{aligned}
&= 2\int_0^2 \left[64 - 48x^2 + 12x^4 - x^6\right] dx \\
&= 2\left[64x - 48\left(\frac{x^3}{3}\right) + 12\left(\frac{x^5}{5}\right) - \frac{x^7}{7}\right]_0^2 \\
&= 2\left[64x - 16x^3 + \frac{12x^5}{5} - \frac{x^7}{7}\right]_0^2 \\
&= 2\left(\left[64(2) - 16(2)^3 + \frac{12(2)^5}{5} - \frac{(2)^7}{7}\right] - [0]\right) \\
&= 2\left[\cancel{128 - 128} + \frac{384}{5} - \frac{128}{7}\right] \\
&= 2\left[\frac{2688}{35} - \frac{640}{35}\right] \\
&= 2\left[\frac{2048}{35}\right] \\
&= \frac{4096}{35} \quad [\text{distance-mass units}]
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\bar{y} &= \frac{M_x}{m} \\
&= \frac{\frac{4096}{35}}{\frac{256}{5}} \\
&= \frac{4096}{35} \cdot \frac{5}{256} \\
&= \frac{16}{7} \quad [\text{distance units}]
\end{aligned}
$$

The center of mass is: $(\bar{x}, \bar{y}) = \boxed{\left(0, \frac{16}{7}\right)}$. Note: $\frac{16}{7} \approx 2.286$.

Note: Although the shape of the region favors lower y-values, the density function favors higher y-values.

4) The point P has cylindrical coordinates $\left(r = 7,\ \theta = \dfrac{\pi}{3},\ z = 2\right)$.

Find the Cartesian coordinates of point P. Fill in the blanks below.
Give exact answers; do not approximate. (9 points)

$$\boxed{\begin{aligned} x &= \underline{\ \frac{7}{2}\ } \\ y &= \underline{\ \frac{7\sqrt{3}}{2}\ } \\ z &= \underline{\ 2\ } \end{aligned}}$$

$$\begin{aligned} x &= r\cos\theta \\ &= 7\cos\left(\frac{\pi}{3}\right) \\ &= 7\left(\frac{1}{2}\right) \\ &= \frac{7}{2} \end{aligned}$$

$$\begin{aligned} y &= r\sin\theta \\ &= 7\sin\left(\frac{\pi}{3}\right) \\ &= 7\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{7\sqrt{3}}{2} \end{aligned}$$

$$z = 2$$

(The Cartesian z is the same as the cylindrical z.)

5) The point Q has spherical coordinates $\left(\rho = 3,\ \phi = \dfrac{\pi}{3},\ \theta = \dfrac{\pi}{6}\right)$.

Find the Cartesian coordinates of point Q. Fill in the blanks below.
Give exact answers; do not approximate. (12 points)

$x = \underline{\dfrac{9}{4}}$

$y = \underline{\dfrac{3\sqrt{3}}{4}}$

$z = \underline{\dfrac{3}{2}}$

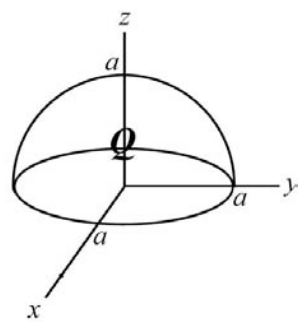
$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ &= 3\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) \\ &= 3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{9}{4} \end{aligned}$$

$$\begin{aligned} y &= \rho \sin\phi \sin\theta \\ &= 3\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right) \\ &= 3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) \\ &= \frac{3\sqrt{3}}{4} \end{aligned}$$

$$\begin{aligned} z &= \rho \cos\phi \\ &= 3\cos\left(\frac{\pi}{3}\right) \\ &= 3\left(\frac{1}{2}\right) \\ &= \frac{3}{2} \end{aligned}$$

6) Find the **mass** of a solid hemisphere of radius *a* lying on its base in terms of the constant of proportionality, *k*. Assume that the mass density at any point in the solid is directly proportional to the distance from the center of the base to the point. Use spherical coordinates to complete the problem. (18 points)

Orient the hemisphere so that its base lies in the *xy*-plane and the center of the base is at the origin.



The density function is given by $\delta(x,y,z) = k\sqrt{x^2+y^2+z^2}$, or $\delta = k\rho$.

Find the mass of the hemisphere (corresponding to region "*Q*"):

$$m = \iiint_Q \delta(x,y,z)\,dV$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/2}\int_{\rho=0}^{\rho=a} \overbrace{k\rho}^{\delta}\cdot\overbrace{\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}^{dV}$$

We can "separate" the above by variable.

$$= k\left[\int_{\theta=0}^{\theta=2\pi} d\theta\right]\left[\int_{\phi=0}^{\phi=\pi/2}\sin\phi\,d\phi\right]\left[\int_{\rho=0}^{\rho=a}\rho^3\,d\rho\right]$$

$$= k\left(\left[\theta\right]_0^{2\pi}\right)\left(\left[-\cos\phi\right]_0^{\pi/2}\right)\left(\left[\frac{\rho^4}{4}\right]_0^a\right)$$

$$= k(2\pi)\left(\left[-\cos\frac{\pi}{2}\right]-\left[-\cos 0\right]\right)\left(\frac{a^4}{4}\right)$$

$$= k(2\pi)\underbrace{\left(0-(-1)\right)}_{=1}\left(\frac{a^4}{4}\right)$$

$$= \boxed{\frac{\pi a^4 k}{2}\quad(\text{mass units})}$$

7) Use the Jacobian-based change of variables method to evaluate $\iint\limits_R \left(9x^2+16y^2\right)dA$, where R is the region in the xy-plane bounded by the graph of $9x^2+16y^2=36$. Use the change of variables: $x=\dfrac{u}{3}$ and $y=\dfrac{v}{4}$.

Hint: You will eventually want to go to polar coordinates.

Give a simplified exact answer; you do not have to approximate it. (20 points)

<u>Rewrite the integrand in terms of u and v:</u>

$$\begin{aligned} 9x^2+16y^2 &= 9\left(\frac{u}{3}\right)^2+16\left(\frac{v}{4}\right)^2 \\ &= u^2+v^2 \end{aligned}$$

<u>Find the Jacobian</u>: $\dfrac{\partial(x,y)}{\partial(u,v)}$

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \\ &= \begin{vmatrix} \dfrac{1}{3} & 0 \\ 0 & \dfrac{1}{4} \end{vmatrix} \\ &= \frac{1}{12} \end{aligned}$$

(Remember that the transpose of a square matrix has the same determinant as the original.)

<u>What is the boundary of R in the xy-plane?</u>

It is an ellipse:

$$\begin{aligned} 9x^2+16y^2 &= 36 \\ \frac{9x^2}{36}+\frac{16y^2}{36} &= 1 \\ \frac{x^2}{4}+\frac{y^2}{9/4} &= 1 \end{aligned}$$

Note: The integrand is continuous throughout R.

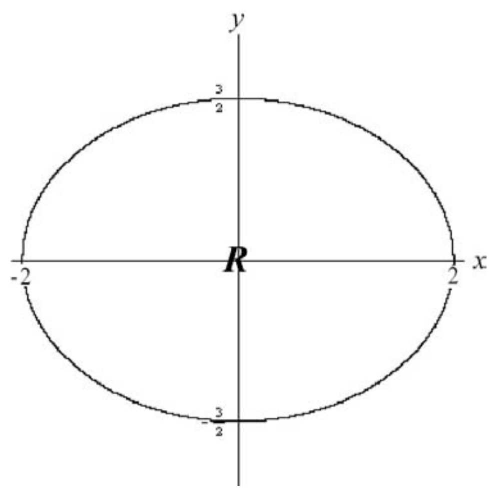
What is the boundary of S in the uv-plane?

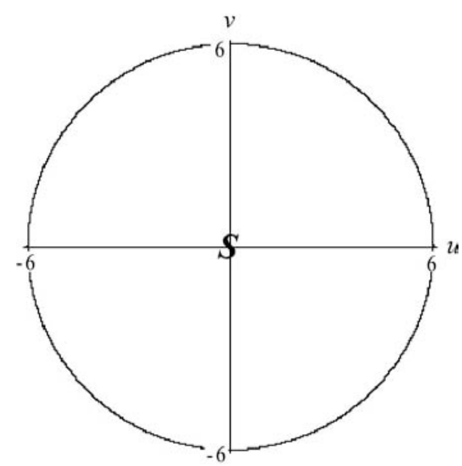
Let S be the region in the uv-plane that R is mapped to under the change of variables transformation.

$$9x^2 + 16y^2 = 36$$
$$9\left(\frac{u}{3}\right)^2 + 16\left(\frac{v}{4}\right)^2 = 36$$
$$u^2 + v^2 = 36$$

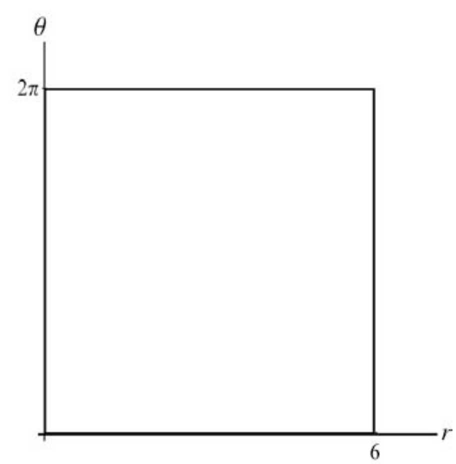
The boundary of our new region of integration, S, will be a circle.

Graphs:





We will eventually go to polar coordinates, which represents another change of variables. We get our favorite kind of region:



Set up the iterated integral and evaluate.

$$\iint_R \left(9x^2+16y^2\right)dA = \iint_S \left(u^2+v^2\right)\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Note: $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{1}{12}\right| = \frac{1}{12}$.

$$= \iint_S \left(u^2+v^2\right)\cdot\frac{1}{12}\cdot du\,dv$$

$$\left(\text{Note: This equals } 4\int_{u=0}^{u=6}\int_{v=0}^{v=\sqrt{36-u^2}} \left(u^2+v^2\right)\cdot\frac{1}{12}\cdot dv\,du\right)$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=6} r^2\cdot\frac{1}{12}\cdot r\,dr\,d\theta$$

$$= \frac{1}{12}\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=6} r^3\,dr\,d\theta$$

$$= \frac{1}{12}\left[\int_{\theta=0}^{\theta=2\pi} d\theta\right]\left[\int_{r=0}^{r=6} r^3\,dr\right]$$

$$= \frac{1}{12}\left[2\pi\right]\left[\frac{r^4}{4}\right]_{r=0}^{r=6}$$

$$= \frac{\pi}{6}\left(\left[\frac{(6)^4}{4}\right]-\left[0\right]\right)$$

$$= \frac{\pi(6)^3}{4}$$

$$= \frac{216\pi}{4}$$

$$= \boxed{54\pi}$$