QUIZ 4 (CHAPTER 17) SOLUTIONS

MATH 252 – FALL 2008 – KUNIYUKI
SCORED OUT OF 125 POINTS ⇒ MULTIPLIED BY 0.84 ⇒ 105% POSSIBLE

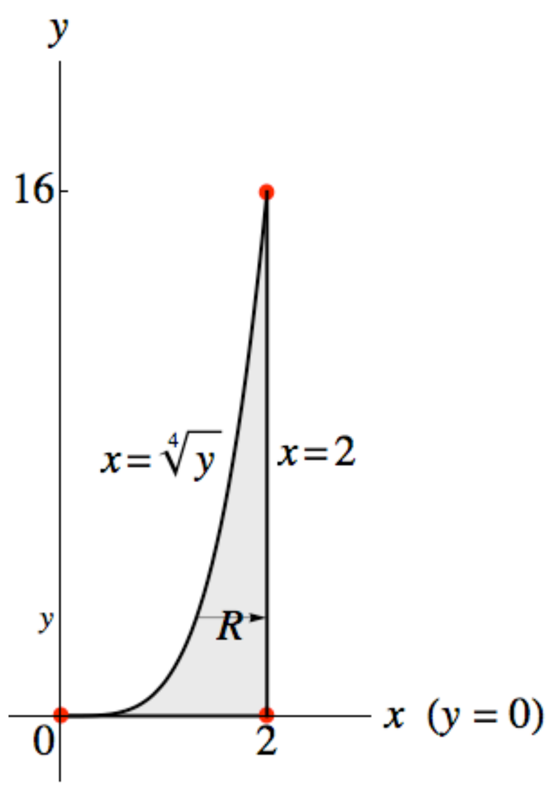
1) Reverse the order of integration, and **evaluate** the resulting double integral: $\int_0^{16}\int_{\sqrt[4]{y}}^{2} \frac{y}{\sqrt{113+x^9}}\,dx\,dy$. Give a simplified exact answer; do not approximate. Sketch the region of integration; you may use different scales for the x- and y-axes. (20 points)

Sketch the region of integration, R.

It may help to label the limits of integration:

$$\text{Let } I = \int_{y=0}^{y=16}\int_{x=\sqrt[4]{y}}^{x=2} \frac{y}{\sqrt{113+x^9}}\,dx\,dy$$

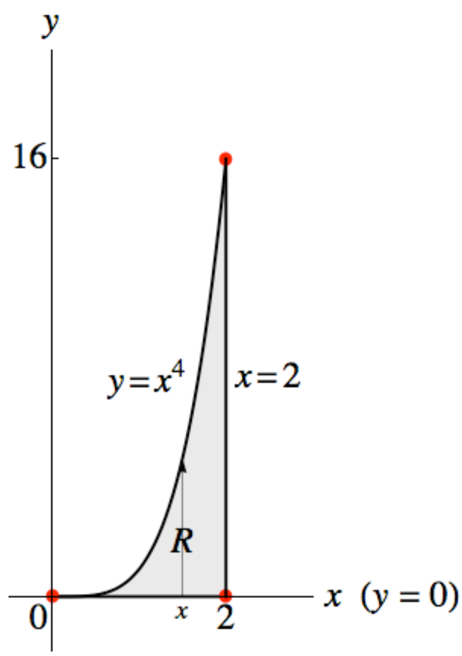
We aim / fix y, shoot x, and slide y.



Reverse the order of integration.

Solve $x = \sqrt[4]{y}$ for y:

$$y = x^4 \quad (x \geq 0)$$



We aim / fix x, shoot y, and slide x.

$$I = \int_{x=0}^{x=2} \int_{y=0}^{y=x^4} \frac{y}{\sqrt{113+x^9}}\, dy\, dx$$

$$= \int_{x=0}^{x=2} \int_{y=0}^{y=x^4} \left(\frac{1}{\sqrt{113+x^9}} \cdot y \right) dy\, dx$$

$$= \int_{x=0}^{x=2} \left(\left[\frac{1}{\sqrt{113+x^9}} \cdot \frac{y^2}{2} \right]_{y=0}^{y=x^4} \right) dx$$

$$= \int_0^2 \left(\left[\frac{1}{\sqrt{113+x^9}} \cdot \frac{\left(x^4\right)^2}{2} \right] - \left[0\right] \right) dx$$

$$= \int_0^2 \frac{x^8}{2\sqrt{113+x^9}}\, dx$$

We now perform a standard u-substitution.

Let $u = 113 + x^9$

$$du = 9x^8\, dx \quad \Rightarrow \quad x^8\, dx = \frac{1}{9} du$$

You can also "Compensate":

$$\int_0^2 \frac{x^8}{2\sqrt{113+x^9}}\, dx = \frac{1}{9} \int_0^2 \frac{9x^8}{2\sqrt{113+x^9}}\, dx$$

Change the limits of integration:

$$x = 0 \quad \Rightarrow \quad u = 113 + \left(0\right)^9 \quad \Rightarrow \quad u = 113$$

$$x = 2 \quad \Rightarrow \quad u = 113 + \left(2\right)^9 \quad \Rightarrow \quad u = 625$$

$$= \frac{1}{9}\int_{u=113}^{u=625} \frac{du}{2\sqrt{u}}$$

$$= \frac{1}{18}\int_{113}^{625} u^{-1/2}\, du$$

$$= \frac{1}{18}\left[\frac{u^{1/2}}{1/2}\right]_{113}^{625}$$

$$= \frac{1}{18}\left[2\sqrt{u}\right]_{113}^{625}$$

$$= \frac{1}{9}\left[\sqrt{u}\right]_{113}^{625}$$

$$= \frac{1}{9}\left(\sqrt{625}-\sqrt{113}\right)$$

$$= \boxed{\frac{25-\sqrt{113}}{9}} \qquad (\text{Note: This is about } 1.59665.)$$

2) Evaluate $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} e^{3x^2+3y^2}\, dy\, dx$. Your answer must be exact and simplified. You do not have to approximate it. (16 points)

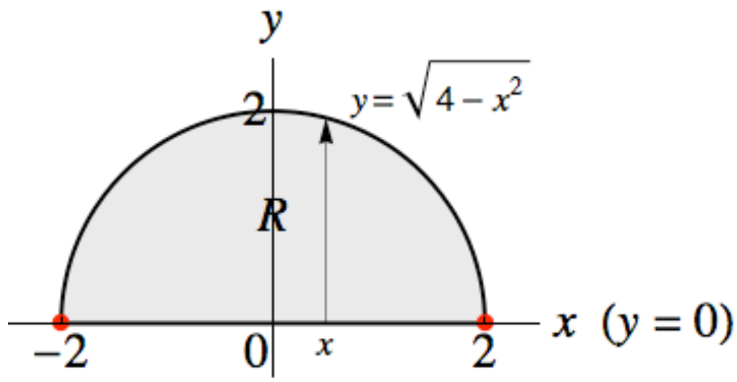
Let's call this double integral "*I*." It may help to label the limits of integration:

$$I = \int_{x=-2}^{x=2}\int_{y=0}^{y=\sqrt{4-x^2}} e^{3x^2+3y^2}\, dy\, dx$$

Sketch the region of integration, R.

If we aim / fix any *x* in $[-2, 2]$, then *y* is shot from $y = 0$ to $y = \sqrt{4-x^2}$.

The graph of $y = \sqrt{4-x^2}$ is a semicircle, the upper half of the circle of radius 2 centered at the origin. Since the *x*-interval is $[-2, 2]$, we do graph the entire semicircle.



Observe that the integrand, $e^{3x^2+3y^2}$, is continuous throughout R.

Convert to polar coordinates.

Why?

- R is well suited for that.
- The integrand is well suited for that.
- It looks like using Cartesian coordinates will be difficult.

$$I = \int_{x=-2}^{x=2} \int_{y=0}^{y=\sqrt{4-x^2}} e^{3x^2+3y^2}\, dy\, dx$$
$$= \iint_R e^{3\left(x^2+y^2\right)}\, dA$$
$$= \iint_R e^{3r^2}\, dA$$
$$= \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2} e^{3r^2} \cdot r\, dr\, d\theta$$

At this point, we can separate the double integral into a product of single integrals. This is because the limits of integration are all constants, and the integrand is separable.

$$= \left[\int_{\theta=0}^{\theta=\pi} d\theta \right] \left[\int_{r=0}^{r=2} re^{3r^2}\, dr \right]$$

Evaluate the first integral:

$$\int_{\theta=0}^{\theta=\pi} d\theta = \left[\theta \right]_{\theta=0}^{\theta=\pi}$$
$$= \pi$$

To evaluate the second integral, $\int_{r=0}^{r=2} re^{3r^2}\, dr$, we perform a standard u-substitution:

Let $\quad u = 3r^2$

$$du = 6r\, dr \quad \Rightarrow \quad r\, dr = \frac{1}{6}\, du$$

You can also “Compensate”:

$$\int_{r=0}^{r=2} re^{3r^2}\, dr = \frac{1}{6} \int_{r=0}^{r=2} 6re^{3r^2}\, dr$$

Change the limits of integration:

$$r = 0 \quad \Rightarrow \quad u = 3(0)^2 \quad \Rightarrow \quad u = 0$$
$$r = 2 \quad \Rightarrow \quad u = 3(2)^2 \quad \Rightarrow \quad u = 12$$

$$I = [\pi]\left[\frac{1}{6}\int_{u=0}^{u=12} e^u\, du\right]$$
$$= \frac{\pi}{6}\int_0^{12} e^u\, du$$
$$= \frac{\pi}{6}\left[e^u\right]_0^{12}$$
$$= \frac{\pi}{6}\left(e^{12} - e^0\right)$$
$$= \boxed{\frac{\pi}{6}\left(e^{12} - 1\right)} \quad (\approx 85{,}218)$$

3) Find the surface area of the portion of the graph of $z = 7x + \frac{1}{2}y^2$ that lies over R, where R is the region in the xy-plane bounded by the rectangle with vertices $(0, 0)$, $(4, 0)$, $(0, 2)$, and $(4, 2)$. Give an exact answer; you do not have to approximate it. Distance is measured in meters.

Major Hint: Use the following Table of Integrals formula:

$$\int \sqrt{a^2 + u^2}\, du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2}\ln\left|u + \sqrt{a^2 + u^2}\right| + C$$

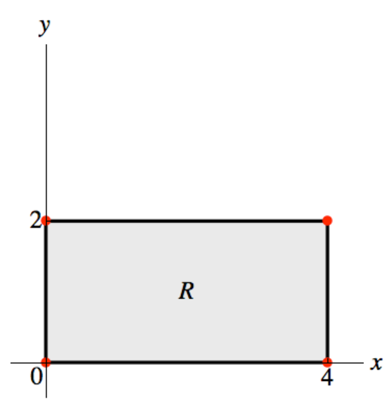
(20 points)

Let S represent the desired surface area.

$$S = \iint_R \sqrt{1 + \left[f_x(x, y)\right]^2 + \left[f_y(x, y)\right]^2}\, dA$$

Let $f(x, y) = 7x + \frac{1}{2}y^2 \quad \Rightarrow \quad \begin{cases} f_x(x, y) = 7, \text{ and} \\ f_y(x, y) = y \end{cases}$

Here's R:



$$S = \int_{x=0}^{x=4} \int_{y=0}^{y=2} \sqrt{1+[7]^2+[y]^2}\, dy\, dx$$
$$= \int_{x=0}^{x=4} \int_{y=0}^{y=2} \sqrt{1+49+y^2}\, dy\, dx$$
$$= \int_{x=0}^{x=4} \int_{y=0}^{y=2} \sqrt{50+y^2}\, dy\, dx$$

At this point, we can separate the double integral into a product of single integrals. Reversing the order of integration would have been easy but unhelpful.

$$= \left[\int_0^2 \sqrt{50+y^2}\, dy\right]\left[\int_0^4 dx\right]$$
$$= \left[\int_0^2 \sqrt{50+y^2}\, dy\right][4]$$

We're going to need a Table of Integrals formula or a trig sub (Section 9.3 in Swokowski's Classic Edition): $y = \sqrt{50}\tan\theta$, or $y = 5\sqrt{2}\tan\theta$.

We use $a^2 = 50$ and $u = y$ in the given Table of Integrals formula.

$$= 4\left[\frac{u}{2}\sqrt{a^2+u^2} \; + \; \frac{a^2}{2}\ln\left|u+\sqrt{a^2+u^2}\,\right|\right]_0^2$$
$$= 4\left[\frac{y}{2}\sqrt{50+y^2} \; + \; \frac{50}{2}\ln\left|y+\sqrt{50+y^2}\,\right|\right]_0^2$$
$$= 2\left[y\sqrt{50+y^2} \; + \; 50\ln\left|y+\sqrt{50+y^2}\,\right|\right]_0^2$$
$$= 2\left(\left[2\sqrt{50+(2)^2}+50\ln\left|2+\sqrt{50+(2)^2}\,\right|\right]-\left[0\sqrt{50+(0)^2}+50\ln\left|0+\sqrt{50+(0)^2}\,\right|\right]\right)$$
$$= 2\left(\left[2\sqrt{54} \; + \; 50\ln\left(2+\sqrt{54}\right)\right] \; - \; \left[0 \; + \; 50\ln\sqrt{50}\right]\right)$$
$$= 2\left(\left[2\left(3\sqrt{6}\right) \; + \; 50\ln\left(2+3\sqrt{6}\right)\right] \; - \; \left[0 \; + \; 50\ln\left(5\sqrt{2}\right)\right]\right)$$

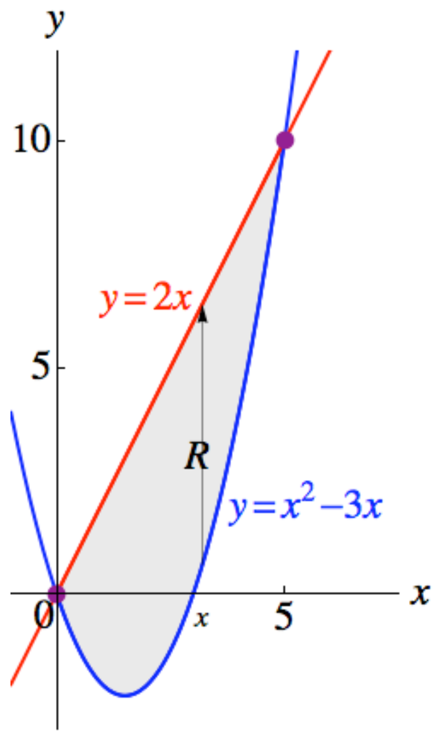
$$= \boxed{\begin{array}{l} 12\sqrt{6} \; + \; 100\ln\left(2+3\sqrt{6}\right) \; - \; 100\ln\left(5\sqrt{2}\right), \text{ or} \\ 4\left(3\sqrt{6} \; + \; 25\ln\left(2+3\sqrt{6}\right) \; - \; 25\ln\left(5\sqrt{2}\right)\right), \text{ or} \\ 4\left[3\sqrt{6} \; + \; 25\ln\left(\dfrac{2+3\sqrt{6}}{5\sqrt{2}}\right)\right], \text{ or} \\ 4\left[3\sqrt{6} \; + \; 25\ln\left(\dfrac{\sqrt{2}+3\sqrt{3}}{5}\right)\right] \text{ square meters} \end{array}}$$

Note: This is about 57.31399 square meters.

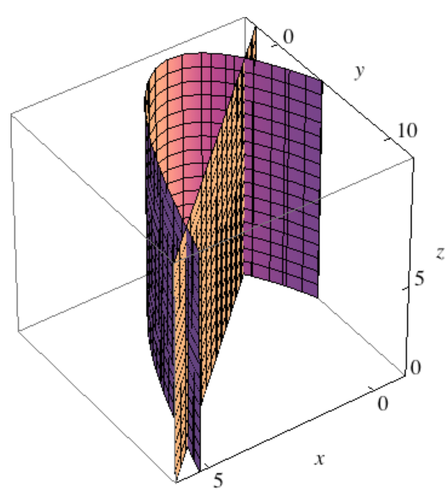
4) **Set up a triple integral** for the volume of the solid bounded by the graphs of $y = x^2 - 3x$, $2x - y = 0$, $z = 5$, and $z - e^x - e^y = 5$. Make sure your triple integral is as detailed as possible; do not leave in generic notation like R or f. Also, do not leave dV in your final answer; break it down into $d(\text{variable})\,d(\text{variable})\,d(\text{variable})$. Do **not** evaluate. (15 points)

Let R, the region of integration, be the projection of this solid on the *xy*-plane. We will sketch R in the *xy*-plane.

The graphs of $y = x^2 - 3x$ (the parabola in blue) and $2x - y = 0$, or equivalently $y = 2x$, (the line in red) are drawn below.



If we were to include *z*-coordinates and look at the 3-D picture, the graph of $y = x^2 - 3x$ would be a parabolic cylinder, and the graph of $y = 2x$ would be a plane.



Find the x-coordinates of the intersection points:

Solve the system $\begin{cases} y = x^2 - 3x \\ y = 2x \end{cases}$ for x.

$$\begin{aligned} x^2 - 3x &= 2x \\ x^2 - 5x &= 0 \\ x(x-5) &= 0 \end{aligned}$$

$$x = 0 \text{ or } x = 5$$

Because the blue parabola opens up, it makes sense that it lies below the red line on the x-interval $[0, 5]$. You could also use the "test value method" to check this out for a value of x strictly between 0 and 5, say 1. (Observe that $x^2 - 3x$ and $2x$ represent continuous functions of x.)

$$\text{Test } x = 1 \quad \Rightarrow \quad y = \left[x^2 - 3x\right]_{x=1} = (1)^2 - 3(1) = -2 \quad (\Rightarrow \text{parabola on bottom})$$

$$\text{Test } x = 1 \quad \Rightarrow \quad y = \left[2x\right]_{x=1} = 2(1) = 2 \quad (\Rightarrow \text{line on top})$$

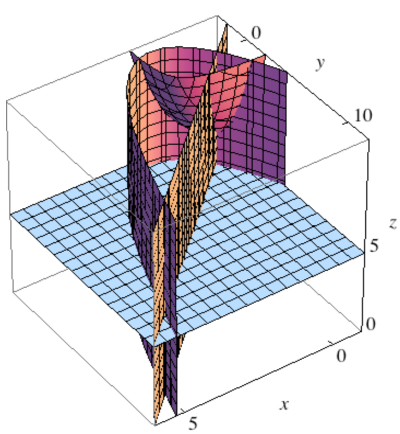
Observe: $z - e^x - e^y = 5 \quad \Leftrightarrow \quad z = 5 + e^x + e^y$. Because $e^x > 0$ and $e^y > 0$ throughout R, it must be true that $z > 5$ throughout R, as well. The graph of $z = 5 + e^x + e^y$ will be the "top" graph relative to the "bottom" graph of $z = 5$. If (x, y) is fixed in R, then z is shot from $z = 5$ up to $z = 5 + e^x + e^y$.

$V = \iiint_Q dV$, where Q is the region represented by the solid, so the desired integral is:

$$\boxed{V = \int_{x=0}^{x=5} \int_{y=x^2-3x}^{y=2x} \int_{z=5}^{z=5+e^x+e^y} dz\, dy\, dx}$$

Below is a peek inside the region whose volume we are setting up.

- The "floor" is part of the graph of $z = 5$, a horizontal plane.
- The "ceiling" is part of the graph of $z = 5 + e^x + e^y$, but the ceiling rises too quickly for us to effectively capture it all.



5) A solid is bounded by the graph of $z = x^2 + y^2$, the graph of $x^2 + y^2 = 16$, and the *xy*-plane. The density at any point in the solid is directly proportional to the square of the distance from the point to the *xy*-plane. Find the center of mass of the solid; it turns out to be a point outside of the actual solid. Use cylindrical coordinates to complete the problem. If you use any short cuts, explain them precisely! Warning: You may run into some very big numbers along the way! (29 points)

The graph of $z = x^2 + y^2$ is a paraboloid opening along the positive *z*-axis.

In cylindrical coordinates, its equation is $z = r^2$.

The graph of $x^2 + y^2 = 16$ is a right circular cylinder of radius 4; the *z*-axis is its axis.

In cylindrical coordinates, its equation is $r^2 = 16$, or $r = 4$.

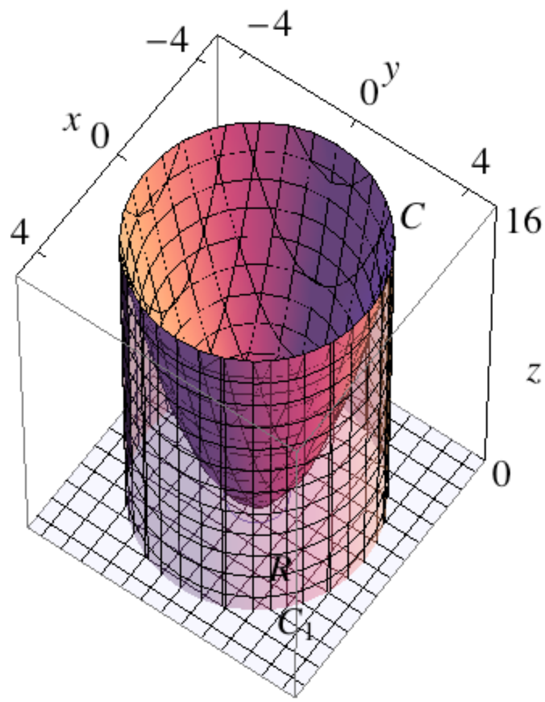
The density function is given by $\delta(x, y, z) = kz^2$.

Let $(\bar{x}, \bar{y}, \bar{z})$ be the center of mass. By symmetry of *Q*, the region corresponding to the solid, about the planes $x = 0$ (i.e., the *yz*-plane) and $y = 0$ (i.e., the *xz*-plane) and the fact that $\delta(x, y, z)$ is even in *x* and *y*, we have $\bar{x} = 0$ and $\bar{y} = 0$.

Find *C*, the circle of intersection between the paraboloid and the cylinder.

$$\begin{cases} z = r^2 \\ r = 4 \end{cases} \Leftrightarrow \begin{cases} z = (4)^2 \\ r = 4 \end{cases} \Leftrightarrow \begin{cases} z = 16 \\ r = 4 \end{cases}$$

The projection of *C* on the *xy*-plane is the circle C_1 of radius 4 in the *xy*-plane centered at the origin. C_1 is the boundary of R, the projection of the solid on the *xy*-plane.



Observe: For a fixed (x, y) in R, the corresponding z-coordinates "shoot" from the plane $z = 0$ (i.e., the xy-plane) up to the paraboloid $z = r^2$.

Find the mass of the solid (corresponding to region "Q"):

$$\begin{aligned}
m &= \iiint_Q \delta(x,y,z)\, dV \\
&= \iiint_Q kz^2\, dV \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=4} \int_{z=0}^{z=r^2} \overbrace{kz^2}^{\delta} \cdot \overbrace{r\, dz\, dr\, d\theta}^{dV} \\
&= k \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=4} \int_{z=0}^{z=r^2} rz^2\, dz\, dr\, d\theta \\
&= k \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=4} \left(r \left[\frac{z^3}{3} \right]_{z=0}^{z=r^2} \right) dr\, d\theta \\
&= \frac{k}{3} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=4} \left(r \left[z^3 \right]_{z=0}^{z=r^2} \right) dr\, d\theta \\
&= \frac{k}{3} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=4} r \left[\left(r^2\right)^3 - (0)^3 \right] dr\, d\theta \\
&= \frac{k}{3} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=4} r \left[r^6 \right] dr\, d\theta \\
&= \frac{k}{3} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=4} r^7\, dr\, d\theta
\end{aligned}$$

At this point, we can separate the double integral into a product of single integrals.

$$\begin{aligned}
&= \frac{k}{3} \left[\int_{r=0}^{r=4} r^7\, dr \right] \left[\int_{\theta=0}^{\theta=2\pi} d\theta \right] \\
&= \frac{k}{3} \left(\left[\frac{r^8}{8} \right]_{r=0}^{r=4} \right) [2\pi] \\
&= \frac{k}{3} \left(\left[\frac{(4)^8}{8} \right] - [0] \right) [2\pi] \\
&= \frac{k}{3} \left(\frac{65536}{8} \right) [2\pi] \\
&= \frac{k}{3} (8192) [2\pi] \\
&= \frac{16{,}384\pi k}{3} \quad [\text{mass units}]
\end{aligned}$$

Find $\bar{z}$.

$$\bar{z} = \frac{M_{xy}}{m}\text{, where:}$$

$$\begin{aligned}
M_{xy} &= \iiint_Q z\,\delta(x,y,z)\,dV \\
&= \iiint_Q z\cdot kz^2\,dV \\
&= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=4}\int_{z=0}^{z=r^2} z\cdot \overbrace{kz^2}^{\delta}\cdot \overbrace{r\,dz\,dr\,d\theta}^{dV} \\
&= k\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=4}\int_{z=0}^{z=r^2} rz^3\,dz\,dr\,d\theta \\
&= k\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=4}\left(r\left[\frac{z^4}{4}\right]_{z=0}^{z=r^2}\right)dr\,d\theta \\
&= \frac{k}{4}\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=4}\left(r\left[z^4\right]_{z=0}^{z=r^2}\right)dr\,d\theta \\
&= \frac{k}{4}\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=4} r\left[\left(r^2\right)^4-(0)^4\right]dr\,d\theta \\
&= \frac{k}{4}\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=4} r\left[r^8\right]dr\,d\theta \\
&= \frac{k}{4}\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=4} r^9\,dr\,d\theta
\end{aligned}$$

At this point, we can separate the double integral into a product of single integrals.

$$\begin{aligned}
&= \frac{k}{4}\left[\int_{r=0}^{r=4} r^9\,dr\right]\left[\int_{\theta=0}^{\theta=2\pi} d\theta\right] \\
&= \frac{k}{4}\left(\left[\frac{r^{10}}{10}\right]_{r=0}^{r=4}\right)\left[2\pi\right] \\
&= \frac{k}{4}\left(\left[\frac{(4)^{10}}{10}\right]-\left[0\right]\right)\left[2\pi\right] \\
&= \frac{k}{4}\left[\frac{(4)^{10}}{{}_5\cancel{10}}\right]\left[\cancel{2}\pi\right] \\
&= k\left[\frac{(4)^9}{5}\right]\left[\pi\right] \\
&= \frac{262{,}144\pi k}{5}\quad\left[\text{distance-mass units}\right]
\end{aligned}$$

Thus,

$$
\begin{aligned}
\bar{z} &= \frac{M_{xy}}{m} \\
&= \frac{\frac{262{,}144\pi k}{5}}{\frac{16{,}384\pi k}{3}} \\
&= \left(\frac{262{,}144\cancel{\pi k}}{5}\right)\left(\frac{3}{16{,}384\cancel{\pi k}}\right) \\
&= \left(\frac{262{,}144}{16{,}384}\right)\left(\frac{3}{5}\right) \\
&= (16)\left(\frac{3}{5}\right) \\
&= \frac{48}{5}\ [\text{distance units}]
\end{aligned}
$$

The center of mass is: $\left(\bar{x}, \bar{y}, \bar{z}\right) = \boxed{\left(0, 0, \frac{48}{5}\right), \text{ or } \left(0, 0, 9.6\right)}$

Note 1: Observe that this point actually lies outside of the solid. It lies 60% of the way along the z-axis from the xy-plane to the plane $z = 16$, which contains the top of the solid.

Note 2: Although the shape of the region favors lower z-values, the density function favors higher z-values.

6) Consider the Cartesian (or rectangular) coordinates *x*, *y*, and *z*, and the spherical coordinates ρ, ϕ, and θ. Verify that, for the spherical coordinate transformation, the Jacobian $\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2 \sin\phi$. Begin by expressing *x*, *y*, and *z* in terms of ρ, ϕ, and θ; you may do this from memory without showing work. Work out an appropriate determinant, and clearly show each step; do not simply use a geometric argument. (25 points)

<u>The spherical coordinate transformation:</u>

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

<u>Find the desired Jacobian:</u>

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial\rho} & \dfrac{\partial y}{\partial\rho} & \dfrac{\partial z}{\partial\rho} \\ \dfrac{\partial x}{\partial\phi} & \dfrac{\partial y}{\partial\phi} & \dfrac{\partial z}{\partial\phi} \\ \dfrac{\partial x}{\partial\theta} & \dfrac{\partial y}{\partial\theta} & \dfrac{\partial z}{\partial\theta} \end{vmatrix} \quad (\text{or the determinant of the transpose})$$

$$= \begin{vmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \rho\cos\phi\cos\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi \\ -\rho\sin\phi\sin\theta & \rho\sin\phi\cos\theta & 0 \end{vmatrix}$$

Let's expand this determinant along the third row so that we can exploit the "0."

$$= +(-\rho\sin\phi\sin\theta)\begin{vmatrix} \sin\phi\sin\theta & \cos\phi \\ \rho\cos\phi\sin\theta & -\rho\sin\phi \end{vmatrix}$$
$$-(\rho\sin\phi\cos\theta)\begin{vmatrix} \sin\phi\cos\theta & \cos\phi \\ \rho\cos\phi\cos\theta & -\rho\sin\phi \end{vmatrix}$$
$$+0$$

$$= (-\rho\sin\phi\sin\theta)(-\rho\sin^2\phi\sin\theta - \rho\cos^2\phi\sin\theta)$$
$$-(\rho\sin\phi\cos\theta)(-\rho\sin^2\phi\cos\theta - \rho\cos^2\phi\cos\theta)$$

$$
\begin{aligned}
&= \left(-\rho\sin\phi\sin\theta\right)\left[\left(-\rho\sin\theta\right)\left(\sin^2\phi+\cos^2\phi\right)\right] \\
&\quad -\left(\rho\sin\phi\cos\theta\right)\left[\left(-\rho\cos\theta\right)\left(\sin^2\phi+\cos^2\phi\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left(-\rho\sin\phi\sin\theta\right)\left[\left(-\rho\sin\theta\right)\left(1\right)\right] \\
&\quad -\left(\rho\sin\phi\cos\theta\right)\left[\left(-\rho\cos\theta\right)\left(1\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left(-\rho\sin\phi\sin\theta\right)\left(-\rho\sin\theta\right) \\
&\quad -\left(\rho\sin\phi\cos\theta\right)\left(-\rho\cos\theta\right)
\end{aligned}
$$

$$= \rho^2\sin\phi\sin^2\theta+\rho^2\sin\phi\cos^2\theta$$

$$= \left(\rho^2\sin\phi\right)\left(\sin^2\theta+\cos^2\theta\right)$$

$$= \left(\rho^2\sin\phi\right)\left(1\right)$$

$$= \rho^2\sin\phi$$