

QUIZ 5 (CHAPTER 18)

SOLUTIONS

MATH 252 – FALL 2006 – KUNIYUKI
105 POINTS TOTAL, BUT 100 POINTS = 100%

Show all work, simplify as appropriate, and use “good form and procedure” (as in class).

Box in your final answers!

No notes or books allowed. A scientific calculator is allowed.

- 1) C consists of the curves C_1 and C_2 in the xy -plane. That is, $C = C_1 \cup C_2$.

The curve C_1 is the directed line segment from $(0, 0)$ to $(4, 2)$, and the curve C_2 is the portion of the parabola $x = y^2$ directed from $(4, 2)$ to $(9, 3)$. If the force at (x, y) is $\mathbf{F}(x, y) = \langle 4y^3, 3x \rangle$, find the work done by \mathbf{F} along C .

It is recommended that you write your final answer as a decimal. (25 points)

Parameterize C_1 :

Parametric equations for C_1 (and dx and dy) are given by:

$$\begin{cases} x = 4t & \Rightarrow & dx = 4 dt \\ y = 2t & \Rightarrow & dy = 2 dt \end{cases} \quad (t: 0 \rightarrow 1)$$

This is because the displacement vector from $(0, 0)$ to $(4, 2)$ is given by $\mathbf{v} = \langle 4, 2 \rangle$. We could also use:

$$\begin{cases} x = 2t & \Rightarrow & dx = 2 dt \\ y = t & \Rightarrow & dy = dt \end{cases} \quad (t: 0 \rightarrow 2)$$

Parameterize C_2 :

$$\begin{cases} x = t^2 & \Rightarrow & dx = 2t dt \\ y = t & \Rightarrow & dy = dt \end{cases} \quad (t: 2 \rightarrow 3)$$

Compute the work integral along C_1 :

$$\begin{aligned}\int_{C_1} \mathbf{F} \bullet d\mathbf{r} &= \int_{C_1} \langle 4y^3, 3x \rangle \bullet \langle dx, dy \rangle \\&= \int_{C_1} 4y^3 dx + 3x dy \quad \left(\text{Think: } \int_{C_1} M dx + N dy \right) \\&= \int_0^1 4(2t)^3 (4 dt) + 3(4t)(2 dt) \\&= \int_0^1 128t^3 dt + 24t dt \\&= \left[32t^4 + 12t^2 \right]_0^1 \\&= \left[32(1)^4 + 12(1)^2 \right] - [0] \\&= [32 + 12] \\&= 44\end{aligned}$$

Compute the work integral along C_2 :

$$\begin{aligned}\int_{C_2} \mathbf{F} \bullet d\mathbf{r} &= \int_{C_2} \langle 4y^3, 3x \rangle \bullet \langle dx, dy \rangle \\&= \int_{C_2} 4y^3 dx + 3x dy \\&= \int_2^3 4(t)^3 (2t dt) + 3(t^2)(dt) \\&= \int_2^3 (8t^4 + 3t^2) dt \\&= \left[\frac{8t^5}{5} + t^3 \right]_2^3 \\&= \left[\frac{8(3)^5}{5} + (3)^3 \right] - \left[\frac{8(2)^5}{5} + (2)^3 \right] \\&= \left[\frac{1944}{5} + 27 \right] - \left[\frac{256}{5} + 8 \right] \\&= \frac{1783}{5} \text{ or } 356.6\end{aligned}$$

The total work along C is given by:

$$\begin{aligned}\int_C \mathbf{F} \bullet d\mathbf{r} &= \int_{C_1} \mathbf{F} \bullet d\mathbf{r} + \int_{C_2} \mathbf{F} \bullet d\mathbf{r} \\&= 44 + 356.6 \\&= \boxed{400.6}\end{aligned}$$

- 2) Use the idea of potential functions and the Fundamental Theorem for Line Integrals to show that the following line integral is independent of path in Octant I of xyz -space and to evaluate the integral. Show all work, as we have done in class. Use good form. In particular, indicate independent variables for functions; for example, write $f(x, y, z)$ instead of simply f . Give an exact, simplified answer; do **not** approximate.

$$\int_{(2,1,1)}^{(1,2,3)} (6x - 1) dx + (4e^{2z}) dy + \left(8ye^{2z} + \frac{1}{z} \right) dz$$

(27 points)

This line integral is of the form $\int_{(2,1,1)}^{(1,2,3)} \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = \left\langle 6x - 1, 4e^{2z}, 8ye^{2z} + \frac{1}{z} \right\rangle, \text{ if } \mathbf{F} \text{ is independent of path in Octant I.}$$

Find a potential function f for \mathbf{F} in Octant I.

$$\text{We require } f_x(x, y, z) = 6x - 1$$

(Partially integrate both sides with respect to x .)

$$f(x, y, z) = 3x^2 - x + g(y, z)$$

$$f_y(x, y, z) = g_y(y, z)$$

$$\text{We require } f_y(x, y, z) = 4e^{2z}$$

$$\text{Then, } g_y(y, z) = 4e^{2z}$$

(Partially integrate both sides with respect to y .)

$$\begin{aligned} g(y, z) &= (4e^{2z})y + h(z) \\ &= 4ye^{2z} + h(z) \end{aligned}$$

$$f(x, y, z) = 3x^2 - x + 4ye^{2z} + h(z)$$

Reminder: $f(x, y, z) = 3x^2 - x + 4ye^{2z} + h(z)$

$$\begin{aligned}f_z(x, y, z) &= 4y(2e^{2z}) + h'(z) \\&= 8ye^{2z} + h'(z)\end{aligned}$$

We require $f_z(x, y, z) = 8ye^{2z} + \frac{1}{z}$

$$\text{Then, } h'(z) = \frac{1}{z}$$

(Integrate both sides with respect to z .)

$$h(z) = \ln z + K$$

Note: Assume $z > 0$, since we restrict ourselves to Octant I.

Then, $f(x, y, z) = 3x^2 - x + 4ye^{2z} + \ln z + K$ (We can take $K = 0$.)

This shows that \mathbf{F} is independent of path in Octant I of xyz -space. The above steps were all legitimate under the restriction to Octant I.

Evaluate the given line integral using the Fundamental Theorem for Line Integrals.

$$\begin{aligned}&\int_{(2,1,1)}^{(1,2,3)} \underbrace{(6x-1)dx + (4e^{2z})dy + \left(8ye^{2z} + \frac{1}{z}\right)dz}_{=\mathbf{F}\bullet d\mathbf{r}} \\&= \left[\underbrace{3x^2 - x + 4ye^{2z} + \ln z}_{=f(x,y,z)} \right]_{(2,1,1)}^{(1,2,3)} \\&= \left[3(1)^2 - (1) + 4(2)e^{2(3)} + \ln(3) \right] - \left[3(2)^2 - (2) + 4(1)e^{2(1)} + \ln(1) \right] \\&= \left[3 - 1 + 8e^6 + \ln 3 \right] - \left[12 - 2 + 4e^2 + 0 \right] \\&= \left[2 + 8e^6 + \ln 3 \right] - \left[10 + 4e^2 \right] \\&= 2 + 8e^6 + \ln 3 - 10 - 4e^2 \\&= \boxed{8e^6 - 4e^2 + \ln 3 - 8}\end{aligned}$$

3) Let $\mathbf{F}(x, y, z) = 3\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$. Let S be the portion of the “half” cone $z = \sqrt{x^2 + y^2}$ that is inside the cylinder $x^2 + y^2 = 4$ but outside the cylinder $x^2 + y^2 = 1$. Find the flux of \mathbf{F} across S , given by $\iint_S \mathbf{F} \bullet \mathbf{n} \, dS$, where \mathbf{n} is always taken to be the unit upper normal to S . (25 points)

S is given by $z = f(x, y)$, where f is a nice function outside of $(x = 0, y = 0)$, at least, so we may rewrite the flux integral as we did in the Notes on 18.5:

$$\iint_S \mathbf{F} \bullet \mathbf{n} \, dS = \iint_{R_{xy}} \mathbf{F} \bullet \nabla g \, dA$$

Set up the g function:

$$\underbrace{z - \sqrt{x^2 + y^2}}_{=g(x,y,z)} = 0 \quad \text{or} \quad \underbrace{z - (x^2 + y^2)^{1/2}}_{=g(x,y,z)} = 0$$

Find ∇g :

$$\nabla g = \langle g_x, g_y, g_z \rangle, \text{ where:}$$

$$g_x(x, y, z) = -\frac{1}{2}(x^2 + y^2)^{-1/2}(2x) \\ = -\frac{x}{\sqrt{x^2 + y^2}}$$

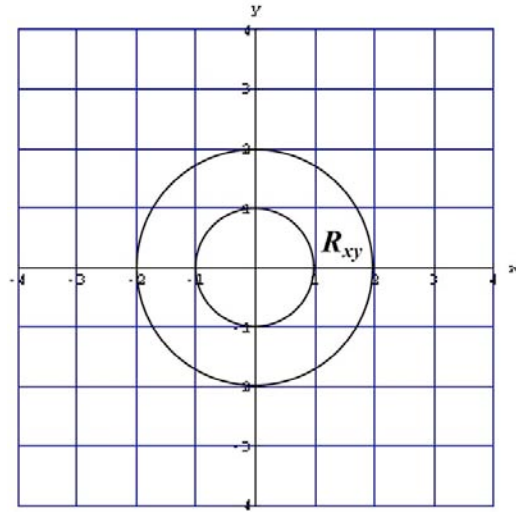
$$g_y(x, y, z) = -\frac{y}{\sqrt{x^2 + y^2}} \text{ by algebraic symmetry of } x \text{ and } y \text{ in } g.$$

$$\text{Then, } \nabla g(x, y, z) = \left\langle -\frac{x}{\sqrt{x^2 + y^2}}, -\frac{y}{\sqrt{x^2 + y^2}}, 1 \right\rangle$$

Express the flux:

$$\begin{aligned} \iint_S \mathbf{F} \bullet \mathbf{n} \, dS &= \iint_{R_{xy}} \mathbf{F} \bullet \nabla g \, dA \\ &= \iint_{R_{xy}} \langle 3, 5, -7 \rangle \bullet \left\langle -\frac{x}{\sqrt{x^2 + y^2}}, -\frac{y}{\sqrt{x^2 + y^2}}, 1 \right\rangle dA \\ &= \iint_{R_{xy}} \left(-\frac{3x}{\sqrt{x^2 + y^2}} - \frac{5y}{\sqrt{x^2 + y^2}} - 7 \right) dA \end{aligned}$$

The projection of S (which is called a “frustum of a cone”) on the xy -plane, R_{xy} , is the annulus (ring) bounded by the following two circles centered at the origin: the circle of radius 2 on the outside, and the smaller circle of radius 1 on the inside.



We will express the flux integral in polar coordinates:

$$\begin{aligned}
 \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} \left(-\frac{3\cancel{r} \cos \theta}{1\cancel{r}} - \frac{5\cancel{r} \sin \theta}{1\cancel{r}} - 7 \right) r \, dr \, d\theta \\
 &= \int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} (-3\cos \theta - 5\sin \theta - 7) r \, dr \, d\theta \\
 &= \left[\int_{\theta=0}^{\theta=2\pi} (-3\cos \theta - 5\sin \theta - 7) d\theta \right] \left[\int_{r=1}^{r=2} r \, dr \right] \\
 &= \left[-3\sin \theta + 5\cos \theta - 7\theta \right]_{\theta=0}^{\theta=2\pi} \left[\frac{r^2}{2} \right]_{r=1}^{r=2} \\
 &= \left(\left[-3\sin(2\pi) + 5\cos(2\pi) - 7(2\pi) \right] - \left[-3\sin(0) + 5\cos(0) - 7(0) \right] \right) \cdot \\
 &\quad \left(\frac{(2)^2}{2} - \frac{(1)^2}{2} \right) \\
 &= \left(\left[-3(0) + 5(1) - 14\pi \right] - \left[-3(0) + 5(1) - 0 \right] \right) \left(\frac{3}{2} \right) \\
 &= \left(\left[\cancel{5} - 14\pi \right] - \left[\cancel{5} \right] \right) \left(\frac{3}{2} \right) \\
 &= -14\pi \left(\frac{3}{2} \right) \\
 &= \boxed{-21\pi}
 \end{aligned}$$

- 4) Use the Divergence Theorem to find the flux of $\mathbf{F}(x, y, z) = \langle \ln(yz), 7y, z \rangle$ through any sphere S of radius 3. (10 points)

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_Q (\text{div } \mathbf{F}) \, dV, \text{ where}$$

$$\begin{aligned} \text{div } \mathbf{F} &= \frac{\partial}{\partial x} [\ln(yz)] + \frac{\partial}{\partial y} (7y) + \frac{\partial}{\partial z} (z) \\ &= 0 + 7 + 1 \\ &= 8 \end{aligned}$$

and Q is the region bounded by S .

$$\begin{aligned} \text{Flux} &= \iiint_Q 8 \, dV \\ &= 8 \iiint_Q dV \\ &= 8(\text{Volume of } Q) \\ &= 8 \left(\frac{4}{3} \pi (3)^3 \right) \end{aligned}$$

since the volume of a sphere of radius r is $\frac{4}{3} \pi r^3$

$$\begin{aligned} &= 8(36\pi) \\ &= \boxed{288\pi} \end{aligned}$$

- 5) Assume that the hypotheses of Stokes's Theorem (as stated in my 18.7 Notes) are satisfied. In particular, S has equation $z = f(x, y)$ and is a "capping surface" for a piecewise smooth simple closed curve C . Fill in the blank: (3 points)

According to Stokes's Theorem, $\oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \underline{\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS}$

Or, given our setup, $\oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \underline{\iint_{R_{xy}} (\text{curl } \mathbf{F}) \cdot \nabla g \, dA}$

6) Let $\mathbf{F}(x, y, z) = \langle y^2 \sin x, 4e^y + y, x^3 yz^3 \rangle$. (15 points total)

a) Find $\text{div } \mathbf{F}$.

$$\begin{aligned}
 \text{div } \mathbf{F} &= \nabla \cdot \mathbf{F} \\
 &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle y^2 \sin x, 4e^y + y, x^3 yz^3 \rangle \\
 &= \frac{\partial}{\partial x}(y^2 \sin x) + \frac{\partial}{\partial y}(4e^y + y) + \frac{\partial}{\partial z}(x^3 yz^3) \\
 &= y^2 \cos x + (4e^y + 1) + x^3 y(3z^2) \\
 &= \boxed{y^2 \cos x + 4e^y + 1 + 3x^3 yz^2}
 \end{aligned}$$

b) Find $\text{curl } \mathbf{F}$.

$$\begin{aligned}
 \text{curl } \mathbf{F} &= \nabla \times \mathbf{F} \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 \sin x & 4e^y + y & x^3 yz^3 \end{vmatrix} \\
 &= \left[\frac{\partial}{\partial y}(x^3 yz^3) - \frac{\partial}{\partial z}(4e^y + y) \right] \mathbf{i} - \left[\frac{\partial}{\partial x}(x^3 yz^3) - \frac{\partial}{\partial z}(y^2 \sin x) \right] \mathbf{j} + \\
 &\quad \left[\frac{\partial}{\partial x}(4e^y + y) - \frac{\partial}{\partial y}(y^2 \sin x) \right] \mathbf{k} \\
 &= [x^3 z^3 - 0] \mathbf{i} - [3x^2 yz^3 - 0] \mathbf{j} + [0 - 2y \sin x] \mathbf{k} \\
 &= \boxed{\begin{aligned} &(x^3 z^3) \mathbf{i} - (3x^2 yz^3) \mathbf{j} - (2y \sin x) \mathbf{k} \\ \text{or } &\langle x^3 z^3, -3x^2 yz^3, -2y \sin x \rangle \end{aligned}}
 \end{aligned}$$