QUIZ 5 (CHAPTER 18) SOLUTIONS

MATH 252 – FALL 2008 – KUNIYUKI
105 POINTS TOTAL, BUT 100 POINTS = 100%

Show all work, simplify as appropriate, and use "good form and procedure" (as in class).
Box in your final answers!
No notes or books allowed. A scientific calculator is allowed.

1) Matching. (9 points total)

Fill in each blank below with a true property describing the vector field **F**. (Assume that we are only evaluating **F** on its domain.)

A. The vectors in the field all have the same direction.
B. The non-**0** vectors in the field all point away from the origin.
C. The vectors in the field are all unit vectors.

I. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$. It is true that __**B**__.

If a point (a,b) is the initial point for $\langle a,b\rangle$, the position vector to the point, then that vector will point away from the origin.

II. $\mathbf{F}(x, y) = 2\mathbf{i} + 3\mathbf{j}$. It is true that __**A**__.

F is a constant vector field.

III. $\mathbf{F}(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}(-x\mathbf{i} - y\mathbf{j})$. It is true that __**C**__.

$$\begin{aligned} \|-x\mathbf{i} - y\mathbf{j}\| &= \sqrt{(-x)^2 + (-y)^2} \\ &= \sqrt{x^2 + y^2} \end{aligned}$$

Thus, $\mathbf{F}(x, y) = \dfrac{-x\mathbf{i} - y\mathbf{j}}{\|-x\mathbf{i} - y\mathbf{j}\|}$. This represents a normalization process.

2) Let $\mathbf{F}(x,y,z) = \left\langle x^2 e^{2z},\ \cos(3y),\ xy^2z^3 - x \right\rangle$. (20 points total)

a) Find div **F**.

$$\begin{aligned}
\text{div } \mathbf{F} &= \nabla \bullet \mathbf{F} \\
&= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \bullet \left\langle x^2e^{2z},\ \cos(3y),\ xy^2z^3 - x \right\rangle \\
&= \frac{\partial}{\partial x}\left(x^2e^{2z}\right) + \frac{\partial}{\partial y}\left(\cos(3y)\right) + \frac{\partial}{\partial z}\left(xy^2z^3 - x\right) \\
&= \left[(2x)\left(e^{2z}\right)\right] + \left[-3\sin(3y)\right] + \left[\left(xy^2\right)\left(3z^2\right)\right] \\
&= \boxed{2xe^{2z} - 3\sin(3y) + 3xy^2z^2}
\end{aligned}$$

b) Find **curl F**.

$$\begin{aligned}
\textbf{curl } \mathbf{F} &= \nabla \times \mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2e^{2z} & \cos(3y) & xy^2z^3 - x \end{vmatrix} \\
&= \left[\frac{\partial}{\partial y}\left(xy^2z^3 - x\right) - \frac{\partial}{\partial z}\left(\cos(3y)\right)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}\left(xy^2z^3 - x\right) - \frac{\partial}{\partial z}\left(x^2e^{2z}\right)\right]\mathbf{j} + \\
&\quad \left[\frac{\partial}{\partial x}\left(\cos(3y)\right) - \frac{\partial}{\partial y}\left(x^2e^{2z}\right)\right]\mathbf{k} \\
&= \left[\left(xz^3\right)(2y) - 0\right]\mathbf{i} - \left[\left(y^2z^3 - 1\right) - \left(x^2\right)\left(2e^{2z}\right)\right]\mathbf{j} + \left[0 - 0\right]\mathbf{k} \\
&= \boxed{\begin{array}{l} \left(2xyz^3\right)\mathbf{i} - \left(y^2z^3 - 1 - 2x^2e^{2z}\right)\mathbf{j} \\ \text{or } \left\langle 2xyz^3,\ -y^2z^3 + 1 + 2x^2e^{2z},\ 0 \right\rangle \\ \text{or } \left\langle 2xyz^3,\ 2x^2e^{2z} - y^2z^3 + 1,\ 0 \right\rangle \end{array}}
\end{aligned}$$

3) C consists of the curves C_1 and C_2 in xyz-space. That is, $C = C_1 \cup C_2$. The curve C_1 is the directed line segment from $(0, 2, 3)$ to $(2, 8, 4)$, and the curve C_2 is the portion of the graph of $y = x^3$ in the plane $z = 4$ directed from $(2, 8, 4)$ to $(3, 27, 4)$. If the force at (x, y, z) is $\mathbf{F}(x, y, z) = \langle xy,\ z+4,\ 3\rangle$, find the work done by $\mathbf{F}$ along C. It is recommended that you write your final answer as a decimal. Hint: If you want, you can analyze C_2 first. (32 points)

Parameterize C_1:

The displacement vector from $(0, 2, 3)$ to $(2, 8, 4)$ is given by:

$$\begin{aligned}\mathbf{v} &= \langle 2-0,\ 8-2,\ 4-3\rangle \\ &= \langle 2, 6, 1\rangle\end{aligned}$$

Parametric equations for C_1 (and dx, dy and dz) are given by:

$$\begin{cases} x = 2t & \Rightarrow \quad dx = 2\,dt \\ y = 2+6t & \Rightarrow \quad dy = 6\,dt \\ z = 3+t & \Rightarrow \quad dz = dt \end{cases} \qquad (t: 0 \to 1)$$

Parameterize C_2:

$$\begin{cases} x = t & \Rightarrow \quad dx = dt \\ y = t^3 & \Rightarrow \quad dy = 3t^2\,dt \\ z = 4 & \Rightarrow \quad dz = 0 \end{cases} \qquad (t: 2 \to 3)$$

Compute the work integral along C_1:

$$\begin{aligned}\int_{C_1} \mathbf{F} \bullet d\mathbf{r} &= \int_{C_1} \langle xy,\ z+4,\ 3\rangle \bullet \langle dx, dy, dz\rangle \\ &= \int_{C_1} xy\,dx \ + \ (z+4)\,dy \ + \ 3\,dz \qquad \left(\text{Think: } \int_{C_1} M\,dx + N\,dy + P\,dz\right) \\ &= \int_0^1 (2t)(2+6t)(2\,dt) \ + \ ((3+t)+4)(6\,dt) \ + \ 3(dt) \\ &= \int_0^1 (8t + 24t^2)\,dt \ + \ (42+6t)\,dt \ + \ 3\,dt \\ &= \int_0^1 (24t^2 + 14t + 45)\,dt \\ &= \left[24\left(\frac{t^3}{3}\right) + 14\left(\frac{t^2}{2}\right) + 45t\right]_0^1\end{aligned}$$

$$= \left[8t^3 + 7t^2 + 45t\right]_0^1$$

$$= \left[8(1)^3 + 7(1)^2 + 45(1)\right] - [0]$$

$$= [8 + 7 + 45]$$

$$= 60$$

Compute the work integral along C_2:

$$\int_{C_2} \mathbf{F} \bullet d\mathbf{r} = \int_{C_2} \langle xy,\, z+4,\, 3\rangle \bullet \langle dx,\, dy,\, dz\rangle$$

$$= \int_{C_2} xy\,dx \;+\; (z+4)\,dy \;+\; 3\,dz \qquad \left(\text{Think: } \int_{C_1} M\,dx + N\,dy + P\,dz\right)$$

$$= \int_2^3 (t)(t^3)(dt) \;+\; ((4)+4)(3t^2\,dt) \;+\; 3(0)$$

$$= \int_2^3 (t^4\,dt) \;+\; (24t^2\,dt)$$

$$= \int_2^3 (t^4 + 24t^2)\,dt$$

$$= \left[\frac{t^5}{5} + 24\left(\frac{t^3}{3}\right)\right]_2^3$$

$$= \left[\frac{t^5}{5} + 8t^3\right]_2^3$$

$$= \left[\frac{(3)^5}{5} + 8(3)^3\right] - \left[\frac{(2)^5}{5} + 8(2)^3\right]$$

$$= [48.6 + 216] - [6.4 + 64]$$

$$= 264.6 - 70.4$$

$$= 194.2 \text{ or } \frac{971}{5}$$

The total work along C is given by:

$$\int_{C} \mathbf{F} \bullet d\mathbf{r} = \int_{C_1} \mathbf{F} \bullet d\mathbf{r} \;+\; \int_{C_2} \mathbf{F} \bullet d\mathbf{r}$$

$$= 60 \;+\; 194.2$$

$$= \boxed{254.2 \text{ or } \frac{1271}{5} \;(\text{work units})}$$

4) Find the exact mass of a thin wire C in *xyz*-space if the density at any point (x, y, z) where $x \geq 0$ is given by $\delta(x, y, z) = 5x$ (i.e., five times the point's distance from the *yz*-plane), and if C is parameterized by $x = 3\cos t$, $y = 3\sin t$, and $y = 7t$, where $0 \leq t \leq \dfrac{\pi}{4}$. (17 points)

$$\begin{aligned}
\text{mass, } m &= \int_C \delta(x, y, z)\, ds \\
&= \int_C 5x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\
&= \int_0^{\frac{\pi}{4}} 5(3\cos t)\sqrt{(-3\sin t)^2 + (3\cos t)^2 + (7)^2}\, dt \\
&= \int_0^{\frac{\pi}{4}} (15\cos t)\sqrt{9\sin^2 t + 9\cos^2 t + 49}\, dt \\
&= \int_0^{\frac{\pi}{4}} (15\cos t)\sqrt{9\Big(\underbrace{\sin^2 t + \cos^2 t}_{=1}\Big) + 49}\, dt \\
&= \int_0^{\frac{\pi}{4}} (15\cos t)\sqrt{58}\, dt \\
&= 15\sqrt{58}\int_0^{\frac{\pi}{4}} \cos t\, dt \\
&= 15\sqrt{58}\left[\sin t\right]_0^{\frac{\pi}{4}} \\
&= 15\sqrt{58}\left(\sin\frac{\pi}{4} - \sin 0\right) \\
&= 15\sqrt{58}\left(\frac{\sqrt{2}}{2} - 0\right) \\
&= 15\sqrt{58}\left(\frac{\sqrt{2}}{2}\right) \\
&= 15\sqrt{29}\cdot\cancel{\sqrt{2}\cdot\frac{\sqrt{2}}{2}}^{\,1} \\
&= \boxed{15\sqrt{29}\quad (\text{mass units})}
\end{aligned}$$

Note: $15\sqrt{29} \approx 80.7775$

5) Use the idea of potential functions and the Fundamental Theorem for Line Integrals to show that the following line integral is independent of path in *xyz*-space and to evaluate the integral. Show all work, as we have done in class. Use good form. In particular, indicate independent variables for functions; for example, write $f(x, y, z)$ instead of simply f. Give an exact, simplified answer; do **not** approximate.

$$\int_{(-1,0,2)}^{(3,4,1)} \left(4y^2 + z\right) dx + \left(8xy - 3ze^{3y}\right) dy + \left(x - e^{3y} + 3z^2\right) dz$$

(27 points)

This line integral is of the form $\int_{(-1,0,2)}^{(3,4,1)} \mathbf{F} \bullet d\mathbf{r}$, where $\mathbf{F}(x,y,z) = \left\langle 4y^2 + z,\ 8xy - 3ze^{3y},\ x - e^{3y} + 3z^2 \right\rangle$, if **F** is independent of path in *xyz*-space.

Find a potential function f for **F** in *xyz*-space.

Compare:

$$f_x(x,y,z) = 4y^2 + z$$

Integrate: Partially integrate both sides with respect to *x*.

$$f(x,y,z) = 4xy^2 + xz + g(y,z)$$

Differentiate: Differentiate both sides with respect to *y*.

$$f_y(x,y,z) = (4x)(2y) + g_y(y,z)$$
$$= 8xy + g_y(y,z)$$

Compare:

We require $f_y(x,y,z) = 8xy - 3ze^{3y}$

Then, $g_y(y,z) = -3ze^{3y}$

Integrate: Partially integrate both sides with respect to *y*.

$$g(y,z) = (-3z)\left(\frac{1}{3}e^{3y}\right) + h(z)$$
$$= -ze^{3y} + h(z)$$

Rewrite:

$$f(x,y,z) = 4xy^2 + xz - ze^{3y} + h(z)$$

Differentiate: Differentiate both sides with respect to *z*.

$$f_z(x,y,z) = x - e^{3y} + h'(z)$$

Compare:

We require $f_z(x,y,z) = x - e^{3y} + 3z^2$

Then, $h'(z) = 3z^2$

Integrate: Integrate both sides with respect to *z*.

$$h(z) = z^3 + K$$

Rewrite:

$$f(x,y,z) = 4xy^2 + xz - ze^{3y} + z^3 + K$$

We can take $K = 0$.

The existence of a potential shows that **F** is independent of path throughout *xyz*-space.

Evaluate the given line integral using the Fundamental Theorem for Line Integrals.

$$\int_{(-1,0,2)}^{(3,4,1)} \mathbf{F} \bullet d\mathbf{r}$$

$$= \int_{(-1,0,2)}^{(3,4,1)} \left(4y^2 + z\right) dx + \left(8xy - 3ze^{3y}\right) dy + \left(x - e^{3y} + 3z^2\right) dz$$

$$= \Big[f(x,y,z) \Big]_{(-1,0,2)}^{(3,4,1)}$$

$$= \Big[4xy^2 + xz - ze^{3y} + z^3 \Big]_{(-1,0,2)}^{(3,4,1)}$$

$$= \Big[4(3)(4)^2 + (3)(1) - (1)e^{3(4)} + (1)^3 \Big] - \Big[4(-1)(0)^2 + (-1)(2) - (2)e^{3(0)} + (2)^3 \Big]$$

$$= \Big[192 + 3 - e^{12} + 1 \Big] - \Big[0 - 2 - 2 + 8 \Big]$$

$$= \Big[196 - e^{12} \Big] - \Big[4 \Big]$$

$$= \boxed{192 - e^{12}}$$

Note: $192 - e^{12} \approx -162{,}563$. Forces are really working against us!