SOLUTIONS TO THE FINAL

MATH 252 – FALL 2006 – KUNIYUKI
60 POINTS TOTAL (15 PROBLEMS; 4 POINTS EACH)

No books allowed. An appropriate sheet of notes and a scientific calculator are allowed.

1) What is the geometric definition of the dot product of two vectors **a** and **b** in V_n? Circle one:

   a) $\boxed{\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta}$

   b) $\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \sin\theta$

   c) $\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \tan\theta$

2) Give symmetric equations for the line in *xyz*-space that passes through the point $(7, 4, -2)$ and that has direction vector $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

$$\boxed{\frac{x-7}{3} = \frac{y-4}{-2} = \frac{z+2}{1}}$$

3) The graph of $3x^2 + 4y^2 - z = 0$ in *xyz*-space is … (circle one):

   a) A Cone

   b) An Ellipsoid

   c) $\boxed{\text{An Elliptic Paraboloid}}$

   d) A Hyperbolic Paraboloid

   e) A Hyperboloid of One Sheet

   f) A Hyperboloid of Two Sheets

Think: $z = 3x^2 + 4y^2$ or simply $z = x^2 + y^2$ for identification purposes.

4) A plane curve C is parameterized by $\mathbf{r}$, a smooth vector-valued function of t, from $t = a$ to $t = b$, where $a < b$. The curve does not overlap itself. Which of the following will give you the arc length of C? Circle one:

a) $\int_a^b \left\| \mathbf{r}(t) \right\| dt$ b) $\boxed{\int_a^b \left\| \mathbf{r}'(t) \right\| dt}$ c) $\int_a^b \left\| \mathbf{r}''(t) \right\| dt$

5) True or False: If $\mathbf{v}$ is a "nice" everywhere differentiable vector-valued function of t, $D_t\left[\mathbf{v}(t) \bullet \mathbf{v}(t)\right] = 2\left[\mathbf{v}'(t) \bullet \mathbf{v}(t)\right]$. Circle one:

$\boxed{\text{True}}$ False

$$\begin{aligned} D_t\left[\mathbf{v}(t) \bullet \mathbf{v}(t)\right] &= \left[\mathbf{v}'(t) \bullet \mathbf{v}(t)\right] + \left[\mathbf{v}(t) \bullet \mathbf{v}'(t)\right] \quad (\text{by a Product Rule for VVFs}) \\ &= 2\left[\mathbf{v}'(t) \bullet \mathbf{v}(t)\right] \end{aligned}$$

6) Two of the following are expressions for curvature that we have covered in class. Circle those two, and only those two.

a) $\left\| \dfrac{d\mathbf{r}}{ds} \right\|$ b) $\boxed{\left\| \dfrac{d\mathbf{T}}{ds} \right\|}$ c) $\left\| \dfrac{d\mathbf{N}}{ds} \right\|$

d) $\dfrac{\left\| \mathbf{r}(t) \times \mathbf{r}'(t) \right\|}{\left\| \mathbf{r}(t) \right\|^3}$ e) $\boxed{\dfrac{\left\| \mathbf{r}'(t) \times \mathbf{r}''(t) \right\|}{\left\| \mathbf{r}'(t) \right\|^3}}$

7) Let f be a "nice" differentiable function of x and y. Give the limit definition of $f_y(x, y)$.

$$\boxed{f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}}$$

8) Let S be the graph of $F(x,y,z)=0$, where $\nabla F(x,y,z)$ is continuous. If $\nabla F(2,-1,3)=\langle 6,2,3\rangle$, write an equation for the tangent plane to S at $(2,-1,3)$.

$$\boxed{\begin{array}{rl} 6(x-2)+2(y-(-1))+3(z-3)=0 & \text{or} \\ 6(x-2)+2(y+1)+3(z-3)=0 & \text{or} \\ 6x+2y+3z=19 & \end{array}}$$

9) Let f be a "nice" function of x and y with continuous second-order partial derivatives. Let $D=f_{xx}f_{yy}-(f_{xy})^2$. The point $(-1,-3)$ is a critical point of f where $D=-20$ and $f_{xx}=5$. Which one of the following does f have at $(-1,-3)$? Circle one:

a) A local maximum

b) A local minimum

c) $\boxed{\text{A saddle point}}$

This is because $D<0$ at the critical point.

10) Express dV in cylindrical coordinates.

$$\boxed{dV = r\,dz\,dr\,d\theta}$$

11) Express dV in spherical coordinates.

$$\boxed{dV = \rho^2 \sin\phi\; d\rho\, d\phi\, d\theta}$$

12) If $x = 2u - 3v$ and $y = 3u + 4v$, find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

Hints: The Jacobian is computed using a determinant. Your answer will be a number.

$$\begin{aligned}\frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \\ &= \begin{vmatrix} 2 & -3 \\ 3 & 4 \end{vmatrix} \\ &= (2)(4) - (-3)(3) \\ &= 8 + 9 \\ &= \boxed{17}\end{aligned}$$

(Remember that the transpose of a square matrix has the same determinant as the original.)

13) Consider the work applied by a force $\mathbf{F}(x,y) = \langle M(x,y), N(x,y) \rangle$ on a particle traveling along a curve C. Three of the following formulas are work formulas that we have covered in class. Circle those three.

a) $\boxed{\int_C M\,dx + N\,dy}$ b) $\boxed{\int_C \mathbf{F} \bullet \mathbf{r}'(t)\,dt}$ c) $\int_C \mathbf{F} \bullet \mathbf{N}\,ds$

d) $\boxed{\int_C \mathbf{F} \bullet \mathbf{T}\,ds}$ e) $\int_C \mathbf{F} \bullet d\mathbf{T}$

14) Which two of the following each guarantees that a vector field $\mathbf{F}$ is conservative throughout $\mathbf{R}^3$? Circle two:

a) If C is any circle in $\mathbf{R}^3$ that encloses the origin, $\oint_C \mathbf{F} \bullet d\mathbf{r} = 0$.

b) $\boxed{\textbf{curl}\,\mathbf{F} = \mathbf{0} \text{ throughout } \mathbf{R}^3.}$

c) $\boxed{\mathbf{F} = \nabla f \text{ for some scalar function } f \text{ throughout } \mathbf{R}^3.}$

15) Assume that the hypotheses of Green's Theorem (as stated in my 18.4 Notes) are satisfied. In particular, C is a piecewise smooth simple closed curve in the *xy*-plane that is the boundary of R, which consists of C and its interior. D is an open region containing R. $\mathbf{F}(x,y) = \langle M(x,y), N(x,y) \rangle$, where *M* and *N* are "nice" in *D*. Fill in the blank:

According to Green's Theorem,

$$\oint_C \mathbf{F} \bullet d\mathbf{r}, \text{ or } \oint_C M\,dx + N\,dy = \boxed{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA} \,.$$