SOLUTIONS TO THE FINAL

MATH 252 – FALL 2008 – KUNIYUKI
60 POINTS TOTAL (15 PROBLEMS; 4 POINTS EACH)

No books allowed. An appropriate sheet of notes and a scientific calculator are allowed.

1) If **a** and **b** are vectors in V_2 such that $\mathbf{a} \bullet \mathbf{b} < 0$, which of the following is true? Box in the best answer:

   a) The angle between **a** and **b** is in $\left[0^\circ, 90^\circ\right)$.

   $\boxed{\text{b) The angle between } \mathbf{a} \text{ and } \mathbf{b} \text{ is in } \left(90^\circ, 180^\circ\right].}$

   c) $\mathbf{a} \bullet \mathbf{b}$ tells us nothing about the angle between **a** and **b**.

$$\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} \text{ and } \mathbf{a} \bullet \mathbf{b} < 0 \quad \Rightarrow$$
$$\cos\theta < 0 \Rightarrow$$
$$\theta \text{ is in } \left(90^\circ, 180^\circ\right].$$

2) Write an equation for the plane in *xyz*-space that is parallel to the plane with equation $3x - 2y + 7z = 10$ and that contains the point $(4, -1, 8)$.

The given plane and the desired plane both have $\langle 3, -2, 7 \rangle$ as a normal vector.

$$\boxed{\begin{aligned} 3(x-4) - 2\left(y - (-1)\right) + 7(z-8) &= 0 \quad \text{or} \\ 3x - 2y + 7z - 70 &= 0 \quad \text{or} \\ 3x - 2y + 7z &= 70 \end{aligned}}$$

3) Consider the graph of $4x^2 - 5y^2 - 6z^2 = 3$ in *xyz*-space. The trace of the graph in the plane $x = 10$ is … (Box in one):

   $\boxed{\text{a) An Ellipse}}$

   b) A Hyperbola

   c) A Parabola

(See next page.)

The trace is given by:

$$\begin{aligned} 4x^2 - 5y^2 - 6z^2 &= 3, & x &= 10 \quad \Rightarrow \\ 4(10)^2 - 5y^2 - 6z^2 &= 3, & x &= 10 \\ 397 &= 5y^2 + 6z^2, & x &= 10 \end{aligned}$$

This is an ellipse.

The ellipse family of traces makes sense, since planes of the form $x = k$ are perpendicular to the axis of the similar graph of $x^2 - y^2 - z^2 = 1$, which, like the given graph, is a hyperboloid of two sheets with the *x*-axis as its axis.

4) If **r** is a vector-valued position function of *t* that is smooth and twice differentiable for all real *t*, which one of the following will be true for all real *t*? Box in one:

a) $\mathbf{r}(t) \perp \mathbf{T}(t)$

We have no reason to believe the Sphere Theorem applies.

b) $\mathbf{r}'(t) \perp \mathbf{T}(t)$

$\boxed{\text{c) } \mathbf{r}'(t) \parallel \mathbf{T}(t)}$

Actually, $\mathbf{r}'$ and **T** point in the same direction for all real *t*, since **T** is the normalized version of $\mathbf{r}'$.

5) If the curvature at a point on a curve (in the real plane) is 10, what is the radius of curvature at that point?

$\boxed{\frac{1}{10}}$, because the radius of curvature at a point *P*, $\rho\Big|_P = \dfrac{1}{\text{curvature } \kappa\Big|_P} = \dfrac{1}{10}$

6) Fill in the blank: The level surface of $f(x, y, z) = 4x^2 + 9y^2 - z^2,\ k = 4$ is __**B**_. (Pick a letter from below.)

A. A Sphere or Ellipsoid
B. A Hyperboloid of One Sheet
C. A Hyperboloid of Two Sheets
D. A Cone
E. A Circular or Elliptic Paraboloid
F. A Hyperbolic Paraboloid
G. A Right Circular or Elliptic Cylinder
H. A Plane
I. A Line (a "degenerate" surface)
J. A Point (a "degenerate" surface)
K. NONE (no surface)

We want the graph of the equation $4 = 4x^2 + 9y^2 - z^2$. Think: $1 = x^2 + y^2 - z^2$ for identification purposes.

7) Assume that f is a differentiable function of *x*, *y*, and *z*. Give the limit definition of $f_z(x, y, z)$.

$$\boxed{f_z(x, y, z) = \lim_{h \to 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}}$$

8) Let S be the graph of $F(x, y, z) = 0$, where $\nabla F(x, y, z)$ is continuous. Assuming that $\nabla F(-2, 4, 1) = \langle 8, 1, -3 \rangle$, write parametric equations for the normal line to *S* at $(-2, 4, 1)$.

$$\boxed{\begin{cases} x = -2 + 8t \\ y = 4 + t \\ z = 1 - 3t \end{cases} \quad (t \in \mathbf{R})}$$

9) Assume that f is a function of x and y with continuous second-order partial derivatives. Let $D = f_{xx} f_{yy} - \left(f_{xy}\right)^2$. The point $(2,-3)$ is a critical point of f where $D = 40$ and $f_{xx} = 2$. Which one of the following does f have at $(2,-3)$? Box in one:

a) A local maximum

$\boxed{\text{b) A local minimum}}$

c) A saddle point

This is because $D > 0$ and $f_{xx} > 0$ (Think: concave up $(\cup)$) at the critical point.

10) The graph of $z = r$ $(r \geq 0)$ in xyz-space is part of … (Box in one):

$\boxed{\text{A cone}}$ A circular cylinder A plane A sphere

See the 17.7 Notes.

11) Express dV in spherical coordinates.

$$\boxed{dV = \rho^2 \sin\phi \; d\rho \, d\phi \, d\theta}$$

12) If $x = 4u + 3v$ and $y = 2u - v$, find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

(Remember that the transpose of a square matrix has the same determinant as the original.)

$$= \begin{vmatrix} 4 & 3 \\ 2 & -1 \end{vmatrix}$$

$$= (4)(-1) - (3)(2)$$

$$= -4 - 6$$

$$= \boxed{-10}$$

13) Consider a vector field **F** that is "nice" throughout $\mathbf{R}^3$; here, "nice" means that the partial derivatives exist for the **i**-, **j**-, and **k**-components of **F** throughout $\mathbf{R}^3$. Assume that, at the origin, $\text{div}\,\mathbf{F} > 0$. Which of the following is at the origin? (Optional Hint: Try to find a "nice" **F** for which $\text{div}\,\mathbf{F} > 0$ throughout $\mathbf{R}^3$, and visualize it.) Box in one:

A sink $\qquad\qquad$ $\boxed{\text{A source}}$

For example, if $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$, then:

$$\begin{aligned} \text{div}\,\mathbf{F} &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \\ &= 1+1+1 \\ &= 3 \quad (> 0 \text{ throughout } \mathbf{R}^3) \end{aligned}$$

Visually: The vectors in the vector field (except the zero vector at the origin) all point away from the origin, so that implies that there is a source at the origin.

14) Consider the work applied by a force $\mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle$ on a particle traveling along a curve *C*. Two of the following formulas are work formulas that we have covered in class. Box in those two.

a) $\int_C \mathbf{F} \bullet \mathbf{r}(t)\, dt$ $\qquad$ b) $\int_C \mathbf{F} \bullet \mathbf{T}'(t)\, dt$ $\qquad$ $\boxed{\text{c) } \int_C \mathbf{F} \bullet \mathbf{T}\, ds}$

$\boxed{\text{d) } \int_C M\, dx + N\, dy}$ $\qquad$ e) $\int_C \mathbf{F} \bullet \mathbf{N}\, ds$

15) If a vector field **F** is continuous in $\mathbf{R}^3$, under what conditions will we be guaranteed that the work integral $\int_C \mathbf{F} \bullet d\mathbf{r} = 0$? Box in one of a), b), or c): (Assume that a circular path corresponds to exactly one revolution.)

a) **F** is conservative in $\mathbf{R}^3$, and *C* is piecewise smooth.

b) $\text{div}\,\mathbf{F} = 0$ throughout $\mathbf{R}^3$, and *C* is a circle in $\mathbf{R}^3$.

$\boxed{\text{c) } \mathbf{F} = \nabla f \text{ throughout } \mathbf{R}^3\text{, where } f(x, y, z) = xyz\text{, and } C \text{ is a circle in } \mathbf{R}^3.}$

If $\mathbf{F} = \nabla f$ for some scalar function f throughout $\mathbf{R}^3$, then **F** is conservative in $\mathbf{R}^3$. A circle is a simple closed curve. If **F** is conservative in $\mathbf{R}^3$, then the work integral along any simple closed curve in $\mathbf{R}^3$ is 0 in value.