MATH 254: NOTES ON 7.1

How do we find eigenvalues for large matrices?

If a matrix is upper or lower triangular, its eigenvalues are simply the entries along the main diagonal.

In Example 6 on pp.386-7, we get relatively lucky with the matrix A. Cofactor expansions can be used to expand $|\lambda I - A|$. If you exploit "0"s along the way, the expansion is quick and easy. It turns out that $|\lambda I - A|$ is simply the product of the diagonal entries of $\lambda I - A$. Of course, we're not always so lucky!

7.1, #21

We will find the eigenvalues of $A = \begin{bmatrix} 0 & -3 & 5 \\ -4 & 4 & -10 \\ 0 & 0 & 4 \end{bmatrix}$.

This problem is similar to Example 8 on p.389. We luck out in that the third row has a couple of "0"s, so we can use it as our "magic row" in our cofactor expansion.

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda & 3 & -5 \\ 4 & \lambda - 4 & 10 \\ 0 & 0 & \lambda - 4 \end{vmatrix} \\ &= +(\lambda - 4)\begin{vmatrix} \lambda & 3 \\ 4 & \lambda - 4 \end{vmatrix} \\ &= (\lambda - 4)\left[\lambda(\lambda - 4) - 12\right] \\ &= (\lambda - 4)\left(\lambda^2 - 4\lambda - 12\right) \\ &= \underbrace{(\lambda - 4)(\lambda - 6)(\lambda + 2)}_{\text{characteristic polynomial}} \end{aligned}$$

The eigenvalues of A are the roots of its characteristic polynomial: 4, 6, and -2.

SEE BACK

7.1, #19

We will find the eigenvalues of $A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$.

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & 2 \\ 2 & \lambda - 5 & 2 \\ 6 & -6 & \lambda + 3 \end{vmatrix}$$

We can expand along the first row.

$$= +(\lambda - 1)\begin{vmatrix} \lambda - 5 & 2 \\ -6 & \lambda + 3 \end{vmatrix} \quad -(-2)\begin{vmatrix} 2 & 2 \\ 6 & \lambda + 3 \end{vmatrix} \quad +(2)\begin{vmatrix} 2 & \lambda - 5 \\ 6 & -6 \end{vmatrix}$$

$$= (\lambda - 1)\left[(\lambda - 5)(\lambda + 3) - (-12)\right] + 2\left[(2)(\lambda + 3) - 12\right] + 2\left[-12 - (6)(\lambda - 5)\right]$$

$$= (\lambda - 1)\left[\lambda^2 - 2\lambda - 15 + 12\right] \quad + 2[2\lambda + 6 - 12] \quad + 2[-12 - 6\lambda + 30]$$

$$= (\lambda - 1)\left[\lambda^2 - 2\lambda - 3\right] \quad + 2[2\lambda - 6] \quad + 2[18 - 6\lambda]$$

Expanding this mess out and combining like terms, we get....

$$= \underbrace{\lambda^3 - 3\lambda^2 - 9\lambda + 27}_{\text{characteristic polynomial}}$$

SHORT WAY

We get lucky with this polynomial, believe it or not!
Factoring by grouping works nicely here….

$$\lambda^3 - 3\lambda^2 - 9\lambda + 27 = \left(\lambda^3 - 3\lambda^2\right) + (-9\lambda + 27)$$

$$= \lambda^2(\lambda - 3) - 9(\lambda - 3)$$

We can now factor out $(\lambda - 3)$.

$$= \underbrace{\left(\lambda^2 - 9\right)}_{\text{Factor}}(\lambda - 3)$$

$$= (\lambda + 3)(\lambda - 3)(\lambda - 3)$$

$$= (\lambda + 3)(\lambda - 3)^2$$

Therefore, -3 is an eigenvalue of multiplicity 1, and 3 is an eigenvalue of multiplicity 2 (which makes a two-dimensional eigenspace possible).

LONG WAY (but more general)

Rational Zero Test, or Rational Roots Theorem

Hopefully, you saw this in Math 141 (Precalculus).

If a polynomial $a_n\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0$ (where the "a_i"s are real coefficients, $a_n \neq 0$, and $a_0 \neq 0$) has rational roots, those roots can be obtained from the form $\pm\frac{p}{q}$, where p is a factor of a_0, and q is a factor of a_n.

Characteristic polynomials are monic (i.e., their leading coefficient, a_n, is always 1), so their rational roots can be obtained from $\pm p$, where p is a factor of a_0, the constant term.

In our example, the characteristic polynomial is $\lambda^3 - 3\lambda^2 - 9\lambda + 27$.
Therefore, any rational roots of this polynomial (i.e., any rational solutions to the characteristic equation $\lambda^3 - 3\lambda^2 - 9\lambda + 27 = 0$) must be in the following list of factors of 27:

$$\pm 1,\ \pm 3,\ \pm 9,\ \pm 27$$

By trial-and-error, it turns out that 3 is a root of the polynomial (plug it in and see!) Therefore, $(\lambda - 3)$ is a factor of the polynomial. We can use long or synthetic division to find the other factor.

Synthetic Division

Root = 3	1	-3	-9	27	← List the coefficients here
	1				← Bring down the "1"

Root = 3	1	-3	-9	27	
		3			← Multiply the "**1**" by the **Root, 3**
	1	**0**			← Add down the column

Root = 3	1	-3	-9	27	
		3	0		← Multiply the "**0**" by the **Root, 3**
	1	**0**	**-9**		← Add down the column

Root = 3	1	-3	-9	27	
		3	0	-27	← Multiply the "**-9**" by the **Root, 3**
	1	**0**	**-9**	0	← Add down the column

The last "0" that we get is our remainder, so there is a clean factorization.
The boldfaced numbers in the bottom row are the coefficients for the quadratic factor we are looking for: $\lambda^2 + 0\lambda - 9$, or simply $\lambda^2 - 9$.

$$\lambda^3 - 3\lambda^2 - 9\lambda + 27 = (\lambda - 3)\underbrace{(\lambda^2 - 9)}_{\text{Factor}}$$

The Quadratic Formula could be used for "worse" quadratics.

$$= (\lambda - 3)(\lambda + 3)(\lambda - 3)$$
$$= (\lambda - 3)^2(\lambda + 3)$$

Again, 3 is an eigenvalue of multiplicity 2, and -3 is an eigenvalue of multiplicity 1.