# FINAL - SOLUTIONS

**MATH 254 - SUMMER 2002 - KUNIYUKI**

**GRADED OUT OF 100 POINTS $\times$ 2 = 200 POINTS TOTAL**

Assume that $n$ represents a positive integer.

1) Find $A^{10}$ if $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. (3 points)

$$\begin{aligned} A^{10} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}^{10} \\ &= \begin{bmatrix} (-1)^{10} & 0 & 0 \\ 0 & (2)^{10} & 0 \\ 0 & 0 & (-2)^{10} \end{bmatrix} \quad \text{(This works, because } A \text{ is a diagonal matrix.)} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1024 & 0 \\ 0 & 0 & 1024 \end{bmatrix} \end{aligned}$$

2) Find the determinant below. (10 points)

$$\begin{vmatrix} 2 & 3 & 3 & 0 & 1 \\ 1 & -1 & -2 & 0 & 4 \\ 0 & 5 & 0 & 0 & 0 \\ 3 & 4 & 1 & 3 & -2 \\ 1 & 0 & 2 & 0 & 1 \end{vmatrix}$$

Let's expand along the third row, since it has many zeros. (The fourth column is also good.) Its only nonzero entry is the "**5**"; the corresponding sign from the sign matrix is "$-$" [or you could observe that $(-1)^{i+j} = (-1)^{3+2} = (-1)^5 = -1$]. To get the submatrix for the corresponding minor, we delete the row and the column containing the "**5**".

$$\begin{vmatrix} 2 & \cancel{3} & 3 & 0 & 1 \\ 1 & \cancel{-1} & -2 & 0 & 4 \\ \cancel{\mathbf{0}} & \cancel{\mathbf{5}} & \cancel{\mathbf{0}} & \cancel{\mathbf{0}} & \cancel{\mathbf{0}} \\ 3 & \cancel{4} & 1 & 3 & -2 \\ 1 & \cancel{0} & 2 & 0 & 1 \end{vmatrix} = \underbrace{(-1)}_{\substack{\text{from} \\ \text{sign} \\ \text{matrix}}} (5) \begin{vmatrix} 2 & 3 & 0 & 1 \\ 1 & -2 & 0 & 4 \\ 3 & 1 & \mathbf{3} & -2 \\ 1 & 2 & 0 & 1 \end{vmatrix}$$

Let's expand along the third column. Its only nonzero entry is the "**3**"; the corresponding sign from the sign matrix is "+" [or you could observe that $(-1)^{i+j} = (-1)^{3+3} = (-1)^6 = +1$].

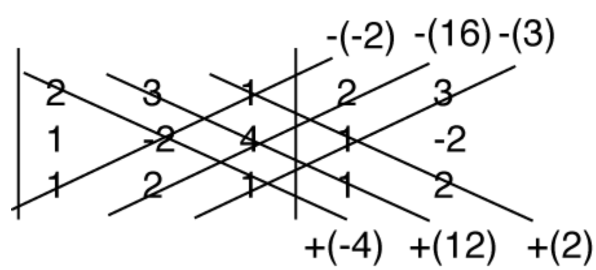
To get the submatrix for the corresponding minor, we delete the row and the column containing the "**3**".

$$= (-5)\begin{vmatrix} 2 & 3 & \cancel{0} & 1 \\ 1 & -2 & \cancel{0} & 4 \\ \cancel{3} & \cancel{1} & \cancel{\mathbf{3}} & \cancel{-2} \\ 1 & 2 & \cancel{0} & 1 \end{vmatrix}$$

$$= (-5)\underbrace{(+1)}_{\substack{\text{from}\\\text{sign}\\\text{matrix}}}(3)\begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 4 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= (-15)\begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 4 \\ 1 & 2 & 1 \end{vmatrix}$$

You could then use Sarrus's Rule or expansion by minors/cofactors to compute the determinant of the resulting $3\times 3$ matrix, which turns out to be $-7$. Using Sarrus's Rule:




Determinant of $3\times 3$ matrix $= 2-16-3-4+12+2=-7$.

$$\text{Original determinant} = (-15)(-7)$$
$$= 105$$

3) If $A$ is a $4\times 4$ matrix, and $|A|=10$, then find $|3A|$. (3 points)

$A$ has order $n = 4$. See Theorem 3.6 on p.130.
(Idea: When you multiply $A$ by 3, you multiply each row by 3; each of these 4 row-multiplications has the effect of multiplying the determinant by 3.)

$$\begin{aligned} |3A| &= 3^n |A| \\ &= 3^4(10) \\ &= (81)(10) \\ &= 810 \end{aligned}$$

4) Let $\mathbf{v}_1 = (0,3,6,0)$, $\mathbf{v}_2 = (0,2,4,6)$, and $\mathbf{v}_3 = (1,-1,-2,1)$.
Express $(4,-1,-2,-11)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.
The weights in this linear combination must be given as specific real numbers; don't just leave them as "$c_i$"s. (12 points)

Solve $\left[\begin{array}{ccc|c} & & & 4 \\ & & & -1 \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & -2 \\ & & & -11 \end{array}\right]$ for $c_1$, $c_2$, and $c_3$, the weights for the linear combination.

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 4 \\ 3 & 2 & -1 & -1 \\ 6 & 4 & -2 & -2 \\ 0 & 6 & 1 & -11 \end{array}\right]$$

Reorder the rows. Wrap around the first row to the bottom.

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -1 \\ 6 & 4 & -2 & -2 \\ 0 & 6 & 1 & -11 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$R_2 + (-2)R_1 \to R_2$.

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 1 & -11 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Reorder the rows. Make the zero row the new bottom row.

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -1 \\ 0 & 6 & 1 & -11 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The matrix is in row-echelon shape. Good enough!

Go further if you want to "officially" use Gaussian or Gauss-Jordan elimination.

$$\begin{cases} 3c_1 + 2c_2 - c_3 = -1 \\ \qquad 6c_2 + c_3 = -11 \\ \qquad\qquad c_3 = 4 \\ \qquad\qquad 0 = 0 \quad \text{(Can ignore)} \end{cases}$$

Back-substitution:

$$c_3 = 4$$

$$\begin{aligned} 6c_2 + c_3 &= -11 \\ 6c_2 + 4 &= -11 \\ 6c_2 &= -15 \\ c_2 &= -\frac{15}{6} \\ c_2 &= -\frac{5}{2} \end{aligned}$$

$$\begin{aligned} 3c_1 + 2c_2 - c_3 &= -1 \\ 3c_1 + 2\left(-\frac{5}{2}\right) - (4) &= -1 \\ 3c_1 - 5 - 4 &= -1 \\ 3c_1 - 9 &= -1 \\ 3c_1 &= 8 \\ c_1 &= \frac{8}{3} \end{aligned}$$

So, $c_1 = \frac{8}{3}$, $c_2 = -\frac{5}{2}$, and $c_3 = 4$.

$$(4, -1, -2, -11) = \frac{8}{3}\mathbf{v}_1 - \frac{5}{2}\mathbf{v}_2 + 4\mathbf{v}_3$$

5) Find the angle between the vectors $(1, 3, 3)$ and $(2, 1, -4)$. Round off your answer to the nearest tenth of a degree. (For maximum accuracy, don't round off until the end.) (8 points)

Label the vectors **v** and **w**, respectively.

$$\begin{aligned} \cos\theta &= \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} \\ &= \frac{(1,3,3) \bullet (2,1,-4)}{\sqrt{(1)^2 + (3)^2 + (3)^2}\sqrt{(2)^2 + (1)^2 + (-4)^2}} \\ &= \frac{2 + 3 - 12}{\sqrt{19}\sqrt{21}} \\ &= \frac{-7}{\sqrt{19}\sqrt{21}} \\ &= -\frac{7}{\sqrt{399}} \quad (\approx -0.3504) \end{aligned}$$

$$\theta \approx 110.5^\circ$$

6) $T: R^4 \to R^3$ is a linear transformation such that $T(\mathbf{v}) = A\mathbf{v}$, where $A$ is the matrix below. (6 points total; 2 points each)

$$A = \begin{bmatrix} \mathbf{1} & 3 & -2 & 2 \\ 0 & 0 & \mathbf{4} & 3 \\ 0 & 0 & 0 & \mathbf{-5} \end{bmatrix}$$

$A$ is already in row-echelon shape. Pivot positions are boldfaced.

a) What is the dimension of the kernel of $T$?

1. (Count the number of nonpivot columns in $A$.)

b) Is $T$ a one-to-one transformation? Circle one:

Yes **No**

Not every column of $A$ has a pivot position.

c) Is $T$ an onto transformation? Circle one:

**Yes** No

Every row of $A$ has a pivot position.

7) $A$ is an $n \times n$ real matrix that has $\lambda$ as a real eigenvalue. Prove that the (real) eigenspace for $\lambda$ is a subspace of $R^n$, as we have done in class. Show all steps! (8 points)

Let $W$ be the eigenspace for $\lambda$.
$W$ must be nonempty, since an eigenvalue has eigenvectors by definition.
$W$ is a subset of $R^n$.

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be any two members of $W$.

Then, $A\mathbf{x}_1 = \lambda\mathbf{x}_1$, and $A\mathbf{x}_2 = \lambda\mathbf{x}_2$ (even if $\mathbf{x}_1$ or $\mathbf{x}_2$ is $\mathbf{0}$).

<u>Prove closure of $W$ under vector addition:</u>

Show $\mathbf{x}_1 + \mathbf{x}_2$ is in $W$ - i.e., $A(\mathbf{x}_1 + \mathbf{x}_2) = \lambda(\mathbf{x}_1 + \mathbf{x}_2)$.

$$\begin{aligned} A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 \\ &= \lambda\mathbf{x}_1 + \lambda\mathbf{x}_2 \\ &= \lambda(\mathbf{x}_1 + \mathbf{x}_2) \end{aligned}$$

So, $\mathbf{x}_1 + \mathbf{x}_2$ is in $W$.

Prove closure of $W$ under scalar multiplication:

Let $c$ be any real scalar.

Show $c\,\mathbf{x}_1$ is in $W$ - i.e., $A(c\,\mathbf{x}_1) = \lambda(c\,\mathbf{x}_1)$.

$$\begin{aligned} A(c\,\mathbf{x}_1) &= c(A\,\mathbf{x}_1) \\ &= c(\lambda\,\mathbf{x}_1) \\ &= \lambda(c\,\mathbf{x}_1) \end{aligned}$$

So, $c\,\mathbf{x}_1$ is in $W$.

Note: You could prove both types of closure by showing that $c\,\mathbf{x}_1 + \mathbf{x}_2$ is in $W$.

QED

8) Orthogonally diagonalize the real symmetric matrix $A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$ by giving an orthogonal matrix $P$ and a diagonal matrix $D$ such that $D = P^{-1}AP$ (or, equivalently, $D = P^T AP$). Hint: The eigenvalues of $A$ are 0 and 5; you don't have to show that. (16 points)

Find two orthogonal unit eigenvectors of $A$:

Find a unit eigenvector for $\lambda_1 = 0$:

Solve the system $[0I - A|\mathbf{0}]$, or $[-A|\mathbf{0}]$.

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right]$$

The reduced row-echelon (RRE) form of this matrix is:

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Now, parametrize the solution set:

$x_1 - 2x_2 = 0 \rightarrow x_1 = 2x_2$
Let $x_2 = t$.

$$\begin{cases} x_1 = 2t \\ x_2 = t \end{cases}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let the eigenvector $\mathbf{p}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

We need our eigenvector to be a unit vector, so let's normalize:

$$\mathbf{u}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

Find a unit eigenvector for $\lambda_2 = 5$:

Solve the system $\left[5I - A \middle| \mathbf{0}\right]$.

$$\left[\begin{array}{cc|c} 5-1 & 0-(-2) & 0 \\ 0-(-2) & 5-4 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right]$$

The reduced row-echelon (RRE) form of this matrix is:

$$\left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Now, parametrize the solution set:

$x_1 + \frac{1}{2}x_2 = 0 \to x_1 = -\frac{1}{2}x_2$

Let $x_2 = t$.

$$\begin{cases} x_1 = -\frac{1}{2}t \\ x_2 = t \end{cases}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

We can rescale $\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ by 2 and get the eigenvector $\mathbf{p}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

<u>Note:</u> You could have observed that we needed $\mathbf{p}_2$ to be orthogonal to $\mathbf{p}_1$ (i.e., $\mathbf{p}_1 \bullet \mathbf{p}_2 = 0$), and you could have figured out an appropriate $\mathbf{p}_2$ based on what we got for $\mathbf{p}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

We need our eigenvector to be a unit vector, so let's normalize:

$$\mathbf{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

Construct the diagonalizing orthogonal matrix, $P$:

$$P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$$
$$= \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

Give $D$:

Let $D = P^{-1}AP$ or, equivalently, $P^T AP$.

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{(The eigenvalues are switched if the columns of } P \text{ are switched.)}$$

9) Rewrite $\dfrac{1}{3-2i}\begin{bmatrix} 4+i & i \\ -2 & 0 \end{bmatrix}$ as an equivalent expression of the form $c\begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}$, where $c$ is a real number, and the "$z_i$"s are complex numbers written in standard form. (8 points)

To rationalize the denominator in $\dfrac{1}{3-2i}$, we need to multiply the numerator and the denominator by the complex conjugate of the denominator.

$$\frac{1}{3-2i}\begin{bmatrix} 4+i & i \\ -2 & 0 \end{bmatrix} = \left(\frac{1}{3-2i}\right)\left(\frac{3+2i}{3+2i}\right)\begin{bmatrix} 4+i & i \\ -2 & 0 \end{bmatrix}$$
$$= \frac{3+2i}{(3)^2+(-2)^2}\begin{bmatrix} 4+i & i \\ -2 & 0 \end{bmatrix} \quad \left(\text{Remember: } (a-bi)(a+bi) = a^2 - b^2i^2 = a^2 + b^2.\right)$$
$$= \frac{3+2i}{13}\begin{bmatrix} 4+i & i \\ -2 & 0 \end{bmatrix}$$

Now, multiply each entry of the matrix by $(3+2i)$.

$$= \frac{1}{13}\begin{bmatrix} (3+2i)(4+i) & (3+2i)(i) \\ (3+2i)(-2) & (3+2i)(0) \end{bmatrix}$$
$$= \frac{1}{13}\begin{bmatrix} 12+3i+8i+\overbrace{2i^2}^{=(-2)} & 3i+\overbrace{2i^2}^{=(-2)} \\ -6-4i & 0 \end{bmatrix}$$
$$= \frac{1}{13}\begin{bmatrix} 10+11i & -2+3i \\ -6-4i & 0 \end{bmatrix}$$

10) Find $\|\mathbf{v}\|$, the Euclidean norm of $\mathbf{v}$, where $\mathbf{v} = (4+i, 3)$ is a vector in $C^2$. (4 points)

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{\mathbf{v} \bullet \mathbf{v}} \\ &= \sqrt{\sum_{i=1}^{2} v_i \overline{v_i}} \\ &= \sqrt{(4+i)(4-i)+(3)(3)} \\ &= \sqrt{16 - \underbrace{i^2}_{=(-1)} + 9} \\ &= \sqrt{16+1+9} \\ &= \sqrt{26}\end{aligned}$$

or

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{|v_1|^2 + |v_2|^2} \\ &= \sqrt{|4+i|^2 + |3|^2} \\ &= \sqrt{17+9} \\ &= \sqrt{26}\end{aligned}$$

11) $A$ is an $n \times n$ real matrix. For each statement below, circle "Yes" if it is equivalent to the statement ***"A* is invertible."** Otherwise, circle "No". You do <u>not</u> have to justify your answers. (10 points total; 2 points each)

a) $\det(A) = 0$. Yes **No**

The correct equivalent statement is "$\det(A) \neq 0$."

b) $A$ is diagonalizable. Yes **No**

For example, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonal (and therefore diagonalizable) but not invertible.

c) $A$ can be written as the product of elementary matrices. **Yes** No

See Theorem 2.14 on p.83.

d) $\operatorname{rank}(A) = n$. **Yes** No

The reduced row-echelon (RRE) form of an $n \times n$ invertible matrix is $I_n$, the $n \times n$ identity matrix. $I_n$ has $n$ pivot positions, so its rank is $n$. Therefore, the rank of $A$ had to be $n$, also. See p.221.

e) $A$ is similar to $I_n$, the $n \times n$ identity matrix. Yes **No**

Only $I_n$ is similar to $I_n$. If $P$ is any invertible $n \times n$ matrix, then $P^{-1} I_n P = P^{-1} P = I_n$. There are many other invertible $n \times n$ matrices aside from $I_n$, however.

**TRUE or FALSE (12 points; 3 points each)**

Circle "True" or "False" (as appropriate) for each statement below.
You do not have to justify your answers.

Remember that you must circle "False" if the statement is "sometimes true, but sometimes false."

Assume that $m$ and $n$ represent positive integers and that all matrices are real.

1) If $A$ and $B$ are $n \times n$ matrices, then $(A+B)(A-B) = A^2 - B^2$.

True **False**

$$\begin{aligned}(A+B)(A-B) &= AA - AB + BA - BB \\ &= A^2 - AB + BA - B^2\end{aligned}$$

It is not always true that $AB = BA$! For a counterexample, find two matrices $A$ and $B$ that do not commute (in a multiplicative sense).

2) If $A$ is an $m \times n$ matrix, then $A^T A$ is symmetric.

**True** False

A matrix is symmetric if and only if it equals its own transpose.
First of all, $A^T A$ is defined, because $A^T$ is $n \times m$, and $A$ is $m \times n$.
Prove that $\left(A^T A\right)^T = A^T A$: $\left(A^T A\right)^T = (A)^T \left(A^T\right)^T = A^T A$.
Remember that the transpose of a product equals the reverse product of transposes.

3) A set of 7 vectors in $R^5$ must span $R^5$.

True **False**

For example, this is not true if six of the vectors are scalar multiples of the seventh.

4) If $A$ is an $m \times n$ matrix, then the set of all (compatible) vectors $\mathbf{b}$ that make $A\mathbf{x} = \mathbf{b}$ consistent is a subspace of $R^m$.

**True** False

In fact, this set is Col($A$), the column space of $A$, which is a subspace of $R^m$.
See Theorem 4.19 on p.220.