MIDTERM 1 - SOLUTIONS

MATH 254 - SUMMER 2002 - KUNIYUKI
CHAPTERS 1, 2, 3
GRADED OUT OF 75 POINTS × 2 = 150 POINTS TOTAL

1) Use either Gaussian elimination with back-substitution or Gauss-Jordan elimination to solve the following system. You must use matrices, as we have done in class. Write your answer as an ordered triple of the form (x_1, x_2, x_3).

$$\begin{cases} 4x_1 - 4x_2 + 14x_3 = 50 \\ 3x_2 + 12x_3 = 90 \\ x_1 - x_2 + 3x_3 = 9 \end{cases}$$

(15 points)

The given equations are already in Standard Form.

Write the corresponding augmented matrix and use elementary row operations (EROs) to try to obtain an upper triangular matrix with "1"s along the main diagonal and "0"s below.

$$\left[\begin{array}{ccc|c} 4 & -4 & 14 & 50 \\ 0 & 3 & 12 & 90 \\ 1 & -1 & 3 & 9 \end{array}\right]$$

We need a "1" in the upper left corner. Let's switch Row 1 and Row 3.

$$R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 9 \\ 0 & 3 & 12 & 90 \\ \mathbf{4} & -4 & 14 & 50 \end{array}\right]$$

Now, turn the boldfaced "**4**" into a "0" by adding (-4) times Row 1 to Row 3.

Old R_3	4	-4	14	50
(-4)•R_1	-4	4	-12	-36
New R_3	**0**	**0**	**2**	**14**

New matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 9 \\ 0 & 3 & 12 & 90 \\ 0 & 0 & 2 & 14 \end{array}\right]$$

We need "1"s along the main diagonal, so divide Row 2 through by 3, and divide Row 3 through by 2.

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 9 \\ 0 & 1 & 4 & 30 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

Write the corresponding system.

$$\begin{cases} x_1 - x_2 + 3x_3 = 9 \\ \quad x_2 + 4x_3 = 30 \\ \qquad x_3 = 7 \end{cases}$$

We have x_3. Now, back-substitute to solve for x_2, then x_1.

$$x_3 = 7$$

$$\begin{aligned} x_2 + 4x_3 &= 30 \\ x_2 + 4(7) &= 30 \\ x_2 + 28 &= 30 \\ x_2 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 9 \\ x_1 - (2) + 3(7) &= 9 \\ x_1 - 2 + 21 &= 9 \\ x_1 + 19 &= 9 \\ x_1 &= -10 \end{aligned}$$

Solution set: $\{(-10, 2, 7)\}$. You could check this in the original system.

Note: If we had used Gauss-Jordan Elimination, we would have continued to apply EROs until we obtained the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 7 \end{array}\right]$$ The solution can then be read off on the right-hand-side.

2) Consider the system

$$\begin{cases} x_1 + 2x_2 - x_4 = 6 \\ x_3 + 3x_4 = 7 \end{cases}$$

(10 points total)

a) Write the solution set of the system in parametric form, as we have done in class.

The augmented matrix is already in reduced row-echelon (RRE) form:

$$\left[\begin{array}{cccc|c} \mathbf{1} & 2 & 0 & -1 & 6 \\ 0 & 0 & \mathbf{1} & 3 & 7 \end{array}\right]$$

Notice that columns 2 and 4 in the coefficient matrix do not have a "leading 1", so x_2 and x_4 are free variables. The system is consistent, so there are infinitely many solutions.

The system:

$$\begin{cases} x_1 + 2x_2 - x_4 = 6 \\ x_3 + 3x_4 = 7 \end{cases}$$

Move the free variables, x_2 and x_4, to the right side.

$$\begin{cases} x_1 = 6 - 2x_2 + x_4 \\ x_3 = 7 - 3x_4 \end{cases}$$

Parametrization: Let $x_2 = t$ and $x_4 = u$.

Solution set in parametric form:

$$\begin{cases} x_1 = 6 - 2t + u \\ x_2 = t \\ x_3 = 7 - 3u \\ x_4 = u \end{cases}$$

where t, u are any real numbers.

b) Use part a) to find two particular solutions to the system.
Note: There are infinitely many possible answers.

Plug in any two pairs of numbers for t and u.
For example:
Plug in $t = 0$ and $u = 0$. We get (6, 0, 7, 0).
Plug in $t = 1$ and $u = 0$. We get (4, 1, 7, 0).

3) Find the matrix product $A^T B$ where $A = \begin{bmatrix} 2 & 0 \\ -1 & 4 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & 2 \\ -1 & 0 & 0 \end{bmatrix}$.

(6 points)

The (i,j) entry of $A^T B$ is the "dot product" of row i of A^T and column j of B.

$$\begin{array}{cc} & \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & 2 \\ -1 & 0 & 0 \end{bmatrix} \\ \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix}}_{A^T} & \begin{bmatrix} \mathbf{2} & \mathbf{-7} & \mathbf{-2} \\ \mathbf{-1} & \mathbf{12} & \mathbf{8} \end{bmatrix} \leftarrow \text{Answer} \end{array}$$

4) Find the inverse of the matrix

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 3 & 0 & 0 \end{bmatrix}$$

(6 points)

$$\left[\begin{array}{ccc|ccc} 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 \\ 3 & 0 & 0 & \multicolumn{3}{c}{\underbrace{0 \quad 0 \quad 1}_{I}} \end{array}\right]$$

Reorder the rows so that the last row becomes the first row.

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 \end{array}\right]$$

Rescale the rows so that we get the 3×3 identity matrix on the left.
Divide the top row by 3, the second row by 2, and the third row by 4.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/4 & 0 \end{array}\right]$$

5) Let A be the matrix $\begin{bmatrix} 2 & 0 & 1 \\ -4 & 1 & -3 \\ 0 & 4 & -1 \end{bmatrix}$. An LU-factorization of A is given by $A = LU$,

where $L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$.

Use this LU-factorization to solve the following system

$$\begin{cases} 2x_1 \qquad\quad + \; x_3 = -13 \\ -4x_1 + x_2 - 3x_3 = 40 \\ \qquad\quad 4x_2 - \; x_3 = 29 \end{cases}$$

(10 points)

Idea:

$$A\mathbf{x} = \mathbf{b}$$

$$L\underbrace{U\mathbf{x}}_{\mathbf{y}} = \mathbf{b}$$

First, solve $L\mathbf{y} = \mathbf{b}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -13 \\ 40 \\ 29 \end{bmatrix}$$

Write the corresponding system:

$$\begin{cases} y_1 & & = -13 \\ -2y_1 + y_2 & & = 40 \\ & 4y_2 + y_3 & = 29 \end{cases}$$

Using forward-substitution, we get:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -13 \\ 14 \\ -27 \end{bmatrix}$$

Second, solve $U\mathbf{x} = \mathbf{y}$.

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -13 \\ 14 \\ -27 \end{bmatrix}$$

Write the corresponding system:

$$\begin{cases} 2x_1 \quad + x_3 = -13 \\ \qquad x_2 - x_3 = 14 \\ \qquad\quad 3x_3 = -27 \end{cases}$$

Using back-substitution, we get our solution vector for the problem:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ -9 \end{bmatrix}$$

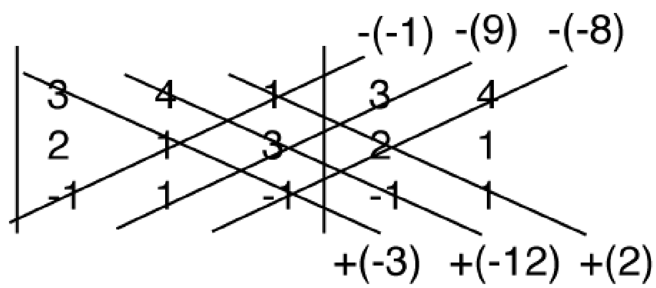
6) Find the determinants. (16 points total)

a) $\begin{vmatrix} 1 & 3 & 4 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ -3 & -1 & 1 & -1 \end{vmatrix}$ (8 points)

Let's expand along the second row, since it has the most zeros.
Its only nonzero entry is the "**2**"; the corresponding sign from the sign matrix is "-" [or you could observe that $(-1)^{i+j} = (-1)^{2+1} = (-1)^3 = -1$]. To get the submatrix for the corresponding minor, we delete the row and the column containing the "**2**".

$$\begin{vmatrix} 1 & 3 & 4 & 1 \\ \mathbf{2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 2 & 1 & 3 \\ -3 & -1 & 1 & -1 \end{vmatrix} = \underbrace{(-1)}_{\text{from sign matrix}} (2) \begin{vmatrix} 3 & 4 & 1 \\ 2 & 1 & 3 \\ -1 & 1 & -1 \end{vmatrix}$$

You could then use Sarrus's Rule or expansion by minors/cofactors to compute the determinant of the resulting 3×3 matrix, which turns out to be -13. Using Sarrus's Rule:



Determinant of 3×3 matrix = 1 – 9 + 8 – 3 – 12 + 2 = -13

$$\begin{aligned} \text{Original determinant} &= -2(-13) \\ &= 26 \end{aligned}$$

b) $\begin{vmatrix} 4 & 3 & 0 & 5 \\ 0 & 0 & 0 & 2 \\ \mathbf{-4} & -3 & 3 & -4 \\ 0 & 2 & 4 & -1 \end{vmatrix}$ (6 points)

Find the determinant by first reducing the matrix to triangular form.

Turn the boldfaced "**-4**" into a "0" by adding Row 1 to Row 3.

Old R_3	-4	-3	3	-4
$+R_1$	4	3	0	5
New R_3	**0**	**0**	**3**	**1**

The row replacement operation does <u>not</u> change the determinant.

$$\begin{vmatrix} 4 & 3 & 0 & 5 \\ 0 & 0 & 0 & 2 \\ -4 & -3 & 3 & -4 \\ 0 & 2 & 4 & -1 \end{vmatrix} = \begin{vmatrix} 4 & 3 & 0 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 2 & 4 & -1 \end{vmatrix}$$

Now, switch Row 2 and Row 4. This will change the sign of the determinant.

$$= -\begin{vmatrix} 4 & 3 & 0 & 5 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

To find the determinant of an upper triangular matrix, take the product of the main diagonal entries.

$$\begin{aligned} &= -(4)(2)(3)(2) \\ &= -48 \end{aligned}$$

c) $\begin{vmatrix} 4 & 3 & 4 \\ 1 & 10 & 1 \\ -7 & 5 & -7 \end{vmatrix}$ (2 points)

The determinant is 0, because two columns are identical.

TRUE or FALSE (12 points; 3 points each)

Circle "True" or "False" (as appropriate) for each statement below.
You do not have to justify your answers.

Remember that you must circle "False" if the statement is "sometimes true, but sometimes false."

Assume that n represents a positive integer.

a) A homogeneous system of three linear equations in five unknowns (variables) must have infinitely many solutions.

True False

A homogeneous system of linear equations must be consistent. If we have more variables than equations, then we end up with a "fat" coefficient matrix, and we must have at least one free variable. Such a system must have infinitely many solutions.

b) If A is an invertible $n \times n$ real matrix, then $A\mathbf{x} = \mathbf{b}$ is a consistent system for any $n \times 1$ real vector $\mathbf{b}$.

True False

This is part of the Invertible Matrix Theorem. In particular, $A\mathbf{x} = \mathbf{b}$ has one unique solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

c) If A and B are invertible $n \times n$ matrices, then $(AB)^{-1} = A^{-1}B^{-1}$.

True **False**

False. The correct statement is: $(AB)^{-1} = B^{-1}A^{-1}$.

d) If A and B are invertible $n \times n$ matrices, then $\det\left(B^{-1}AB\right) = \det(A)$.

True False

$$\begin{aligned} \left|B^{-1}AB\right| &= \left|B^{-1}\right|\left|A\right|\left|B\right| \\ &= \frac{1}{\left|B\right|}\left|A\right|\left|B\right| \end{aligned}$$

The three factors are real numbers.

Note: B is invertible, so $\left|B\right|$ is not 0.

$$= \left|A\right|$$