MIDTERM 2 - SOLUTIONS

MATH 254 - SUMMER 2002 - KUNIYUKI
CHAPTERS 4, 5
GRADED OUT OF 75 POINTS × 2 = 150 POINTS TOTAL

Assume that the "standard" operations for vector addition and scalar multiplication are being used in all relevant problems.

1) For each set below, circle "Yes" if it is a vector space, or circle "No" if it is not.
   - **Whenever you answer "Yes", you do not have to justify your answer.**
   - **Whenever you answer "No", give a counterexample.**

   (10 points total)

a) The set $\left\{(x,y) \,\middle|\, x \text{ is an integer and } y \text{ is a real number}\right\}$ (4 points)

Yes **No**

For example, (1, 0) is in the set, but $\pi(1, 0) = (\pi, 0)$ is not, so the set is not closed under scalar multiplication.

b) The set of all fourth-degree polynomials in x (4 points)

Yes **No**

For example, 0 is not in the set.

c) $\text{Span}\left(\left\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\right\}\right)$, where $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are vectors in R^6 (2 points)

Yes No

The span of a set of vectors in a vector space must be a subspace of the vector space.

2) Let W be the set of all 3×3 real diagonal matrices. Prove that W is a subspace of $M_{3,3}$. You may assume that $M_{3,3}$ is, itself, a vector space without having to prove that. (Completeness and quality are criteria for grading your proof.) (10 points)

W is clearly a nonempty subset of $M_{3,3}$.

Let $\mathbf{v}$ and $\mathbf{w}$ be any two members of W.

$$\mathbf{v} = \begin{bmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{bmatrix}$$, where the diagonal entries are real numbers.

$$\mathbf{w} = \begin{bmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{bmatrix}$$, where the diagonal entries are real numbers.

Prove closure of W under vector addition:

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 & 0 & 0 \\ 0 & v_2 + w_2 & 0 \\ 0 & 0 & v_3 + w_3 \end{bmatrix}.$$

Note that the diagonal entries are real numbers.

$\mathbf{v} + \mathbf{w}$ is in W.

Prove closure of W under scalar multiplication:

Let c be any real scalar.

$$c\,\mathbf{v} = \begin{bmatrix} cv_1 & 0 & 0 \\ 0 & cv_2 & 0 \\ 0 & 0 & cv_3 \end{bmatrix}.$$

Note that the diagonal entries are real numbers.

$c\,\mathbf{v}$ is in W.

Note: You could prove both types of closure by showing that $c\,\mathbf{v} + \mathbf{w}$ is in W.

QED

3) For each of the following sets of vectors, circle "Linearly independent" or "Linearly dependent" as appropriate. You do <u>not</u> have to justify your answers. (12 points; 3 points each)

a) {(1, 2, 7), (3, 6, 19)}

Linearly independent Linearly dependent

We have a set of two vectors, neither of which is a scalar multiple of the other.

b) {(1, 7, 8, 3), (0, 0, 0, 0), (4, -2, 3, 10)}

Linearly independent **Linearly dependent**

Any set with the zero vector must be linearly dependent.

c) $\{1,\ 1+2x,\ 1+2x+3x^2\}$ (Treat these vectors as vectors in P_2.)

Linearly independent Linearly dependent

The naturally corresponding vectors in R^3 are the column vectors in the matrix below:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Notice that the matrix is in row-echelon shape. Each column has a pivot position, so the original set of vectors is linearly independent.

d) The set of four vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, 2\mathbf{v}_1 + \mathbf{v}_2\}$, where $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are vectors in R^7.

Linearly independent **Linearly dependent**

There is a dependency relation among the four vectors.
In particular, if we consider $\mathbf{v}_4 = 2\mathbf{v}_1 + \mathbf{v}_2$, we have that $-2\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_4 = 0$.

4) Does the set of vectors $\{(1,0,4),\ (0,3,0),\ (3,7,12),\ (4,1,16)\}$ span R^3, or not? **Circle** "Spans R^3" or "Does not span R^3" as appropriate, **and justify your answer. Show all work!** (6 points)

Spans R^3 **Does not span R^3**

Stack the given vectors as column vectors in a matrix:

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 3 & 7 & 1 \\ \mathbf{4} & 0 & 12 & 16 \end{bmatrix}$$

Now, turn the boldfaced "**4**" into a "0" by adding (-4) times Row 1 to Row 3.

Old R_3	4	0	12	16
(-4)•R_1	-4	0	-12	-16
New R_3	**0**	**0**	**0**	**0**

New matrix:

$$\begin{bmatrix} \mathbf{1} & 0 & 3 & 4 \\ 0 & \mathbf{3} & 7 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot positions are boldfaced. Not every row has a pivot position, so the original set of vectors fails to span R^3.

5) Let V be the vector space of all 2×2 real symmetric matrices. What is the dimension of V? You do <u>not</u> have to justify your answer. (3 points)

3, because that is the number of vectors in the following basis for V:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

It helps to recognize that V can be represented in the following way:

$$V = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| \ a, b, c \text{ are real numbers} \right\}$$

6) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

By applying elementary row operations (EROs), A can be reduced to the following reduced row-echelon (RRE) form matrix:

$$B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(12 points total)

a) What is the rank of A? (2 points)

3, because that is the number of pivot positions in B.

b) Write a basis for Row(A), the row space of A. (4 points)

We grab the nonzero (pivot) rows from B:
$\{(1, 0, 2, 0, 1),\ (0, 1, -1, 0, 1),\ (0, 0, 0, 1, 1)\}$

c) Write a basis for Col(A), the column space of A. (4 points)

We grab the columns of A that correspond to the pivot columns of B. In other words, we take columns 1, 2, and 4 of A:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ -7 \end{bmatrix} \right\}$$

d) What is the nullity of A? (2 points)

2, because that is the number of nonpivot columns in B.
Also, rank(A) + nullity(A) = n, the total number of columns in A.

$$3 + \text{nullity}(A) = 5$$
$$\text{nullity}(A) = 2$$

7) Let **v** and **w** be two vectors in R^4 such that **v** and **w** are orthogonal, $\| \mathbf{v} \| = 10$, and $\| \mathbf{w} \| = 6$. Find $(\mathbf{v} + 2\mathbf{w}) \bullet (\mathbf{v} - \mathbf{w})$. Your final answer will be a real number. (6 points)

Since **v** and **w** are orthogonal, $\mathbf{v} \bullet \mathbf{w} = 0$, and $\mathbf{w} \bullet \mathbf{v} = 0$.

$$
\begin{aligned}
(\mathbf{v} + 2\mathbf{w}) \bullet (\mathbf{v} - \mathbf{w}) &= \mathbf{v} \bullet \mathbf{v} - \mathbf{v} \bullet \mathbf{w} + 2\mathbf{w} \bullet \mathbf{v} - 2\mathbf{w} \bullet \mathbf{w} \\
&= \| \mathbf{v} \|^2 - 0 + 2(0) - 2\| \mathbf{w} \|^2 \\
&= (10)^2 - 2(6)^2 \\
&= 100 - 72 \\
&= 28
\end{aligned}
$$

8) The set of vectors below is a basis for a two-dimensional subspace of R^4. Use the Gram-Schmidt orthonormalization process to transform this basis

$$
\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ 8 \\ 0 \end{bmatrix} \right\}
$$

into an orthonormal basis for the same subspace.

Hint: The orthogonal projection of a vector **v** onto a vector **w** is given by $\text{proj}_{\mathbf{w}} \mathbf{v} = \left(\dfrac{\mathbf{v} \bullet \mathbf{w}}{\mathbf{w} \bullet \mathbf{w}} \right) \mathbf{w}$.

(16 points)

Label the original set of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively.

We first want a set of orthogonal vectors $\{\mathbf{w}_1, \mathbf{w}_2\}$.

$$
\text{Let } \mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}
$$

Now, let

$$\begin{aligned}
\mathbf{w}_2 &= \mathbf{v}_2 - \operatorname{proj}_{\mathbf{w}_1}\mathbf{v}_2 \\
&= \mathbf{v}_2 - \left(\frac{\mathbf{v}_2 \bullet \mathbf{w}_1}{\mathbf{w}_1 \bullet \mathbf{w}_1}\right)\mathbf{w}_1 \\
&= \begin{bmatrix} -1 \\ 10 \\ 8 \\ 0 \end{bmatrix} - \left(\frac{-1+30+16+0}{1+9+4+1}\right)\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 10 \\ 8 \\ 0 \end{bmatrix} - \left(\frac{45}{15}\right)\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 10 \\ 8 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -4 \\ 1 \\ 2 \\ -3 \end{bmatrix}
\end{aligned}$$

Finally, we want a set of orthonormal vectors $\{\mathbf{u}_1, \mathbf{u}_2\}$. We need to normalize our vectors.

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{(1)^2+(3)^2+(2)^2+(1)^2}}\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{15}}\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{15} \\ 3/\sqrt{15} \\ 2/\sqrt{15} \\ 1/\sqrt{15} \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{\sqrt{(-4)^2+(1)^2+(2)^2+(-3)^2}}\begin{bmatrix} -4 \\ 1 \\ 2 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{30}}\begin{bmatrix} -4 \\ 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -4/\sqrt{30} \\ 1/\sqrt{30} \\ 2/\sqrt{30} \\ -3/\sqrt{30} \end{bmatrix}$$

Our desired orthonormal basis is:

$$\left\{ \begin{bmatrix} 1/\sqrt{15} \\ 3/\sqrt{15} \\ 2/\sqrt{15} \\ 1/\sqrt{15} \end{bmatrix}, \begin{bmatrix} -4/\sqrt{30} \\ 1/\sqrt{30} \\ 2/\sqrt{30} \\ -3/\sqrt{30} \end{bmatrix} \right\}$$