MIDTERM 3 - SOLUTIONS

MATH 254 - SUMMER 2002 - KUNIYUKI
CHAPTERS 6, 7
GRADED OUT OF 75 POINTS × 2 = 150 POINTS TOTAL

1) The linear transformation $T: R^2 \to R^3$ is such that $T(1,1) = (-3,4,1)$ and $T(0,1) = (1,5,3)$. Find $T(4,5)$.
Hint: Remember the definition of a linear transformation.
(5 points)

Express (4,5) as a linear combination of the given basis vectors for R^2, (1,1) and (0,1). Observe that (4,5) = (4,4) + (0,1) = 4(1,1) + (0,1).

$$\begin{aligned} T(4,5) &= T\big(4(1,1)+(0,1)\big) \\ &= 4T(1,1)+T(0,1) \\ &= 4(-3,4,1)+(1,5,3) \\ &= (-12,16,4)+(1,5,3) \\ &= (-11,21,7) \end{aligned}$$

2) The linear transformation $T: R^2 \to R^2$ is such that $T(v_1, v_2) = (2v_1 - v_2\,,\ 2v_1 + 3v_2)$. Find the preimage of $(17,5)$.
(7 points)

We want to solve the system (maybe by using Gauss-Jordan elimination)

$$\begin{cases} 2v_1 - v_2 = 17 \\ 2v_1 + 3v_2 = 5 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & -1 & 17 \\ 2 & 3 & 5 \end{array}\right]$$

Subtract Row 1 from Row 2. $R_2 + (-1)R_1 \to R_2$.

$$\left[\begin{array}{cc|c} 2 & -1 & 17 \\ 0 & 4 & -12 \end{array}\right]$$

Divide Row 2 through by 4.

$$\left[\begin{array}{cc|c} 2 & -1 & 17 \\ 0 & 1 & -3 \end{array}\right]$$

Add Row 2 to Row 1. $R_1 + R_2 \to R_1$.

$$\left[\begin{array}{cc|c} 2 & 0 & 14 \\ 0 & 1 & -3 \end{array}\right]$$

Divide Row 1 through by 2.

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -3 \end{array}\right]$$

The preimage is $\{(7,-3)\}$.

3) The linear transformation $T: R^5 \to R^7$ is such that $\dim(\mathrm{Ker}(T)) = 3$. (11 points total)

a) What is the domain of T? (2 points)

R^5

b) What is $\mathrm{nullity}(T)$? (2 points)

3, since nullity(T) = dim(Ker(T)).

c) What is $\mathrm{rank}(T)$? (3 points)

2, since rank(T) + nullity(T) = dimension of domain = 5.

d) True or False: $\mathrm{Range}(T)$ is a subspace of R^7. Circle one: (2 points)

True False

Note that Range(T) = Col(A), where A is a 7×5 matrix.

e) True or False: $\mathrm{Ker}(T)$ is a subspace of R^7. Circle one: (2 points)

True **False**

Ker(T) is a subspace of the domain, R^5.

4) $T: R^2 \to R^2$ is a linear transformation such that, relative to the standard basis of R^2, $T(x, y) = T(3x - y,\ x + 4y)$. Another basis for R^2 is given by: $B' = \{(1, 3), (2, 0)\}$. (13 points total)

a) Find the standard matrix for *T*. (3 points)

List the components of $T(x,y)$ in rows and line up like terms:

$$\begin{array}{r} 3x - \ \ y \\ x + 4y \end{array}$$

The coefficients then make up *A*, which will be 2×2, since *T* maps R^2 into R^2.

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}$$

b) Find the matrix for *T* relative to the basis B'. (10 points)

The transition matrix from B' to B is:

$$P = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$ (The columns are the B' vectors.)

The transition matrix from B to B' is:

$$\begin{aligned} P^{-1} &= \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}^{-1} \\ &= \frac{1}{\det(P)} \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix} \\ &= \frac{1}{-6} \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix} \\ &= -\frac{1}{6} \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix} \end{aligned}$$

So, the matrix for T relative to B' is:

$$A' = P^{-1}AP$$

$$= -\frac{1}{6}\underbrace{\begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}}_{\begin{bmatrix} -2 & -8 \\ -8 & 7 \end{bmatrix}}\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

$$= -\frac{1}{6}\begin{bmatrix} -26 & -4 \\ 13 & -16 \end{bmatrix} \text{ or } \begin{bmatrix} \frac{13}{3} & \frac{2}{3} \\ -\frac{13}{6} & \frac{8}{3} \end{bmatrix}$$

5) Find the eigenvalues of the following matrix. Show all work!

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 7 \end{bmatrix}$$

(6 points)

Solve $|\lambda I - A| = 0$.

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 2 & 7 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda - 2 & 0 - 3 \\ 0 - 2 & \lambda - 7 \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda - 2 & -3 \\ -2 & \lambda - 7 \end{vmatrix} = 0$$

$$(\lambda - 2)(\lambda - 7) - (-3)(-2) = 0$$

$$\lambda^2 - 9\lambda + 14 - 6 = 0$$

$$\lambda^2 - 9\lambda + 8 = 0$$

$$(\lambda - 1)(\lambda - 8) = 0$$

Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 8$

6) Diagonalize the matrix $A=\begin{bmatrix}1 & 0\\ 6 & -1\end{bmatrix}$ by giving matrices P and D such that $D=P^{-1}AP$, where D is diagonal. You do <u>not</u> have to give P^{-1}.
(16 points)

<u>Find the eigenvalues of A:</u>

A is a lower triangular matrix, so the eigenvalues are on the main diagonal: 1 and -1.

<u>Find two linearly independent eigenvectors of A:</u>

Since we have $n = 2$ distinct real eigenvalues, A is guaranteed to be diagonalizable.

Find an eigenvector for $\lambda_1 = 1$:

Solve the system $\left[1I - A \mid \mathbf{0}\right]$.

$$\left[\begin{array}{cc|c}1-1 & 0-0 & 0\\ 0-6 & 1-(-1) & 0\end{array}\right]$$

$$\left[\begin{array}{cc|c}0 & 0 & 0\\ -6 & 2 & 0\end{array}\right]$$ Now, $R_1 \leftrightarrow R_2$.

$$\left[\begin{array}{cc|c}-6 & 2 & 0\\ 0 & 0 & 0\end{array}\right]$$ Now, divide Row 1 through by (-6).

$$\left[\begin{array}{cc|c}1 & -1/3 & 0\\ 0 & 0 & 0\end{array}\right]$$

$$x_1 - \frac{1}{3}x_2 = 0 \rightarrow x_1 = \frac{1}{3}x_2$$

Let $x_2 = t$.

$$\begin{cases} x_1 = \frac{1}{3}t \\ x_2 = t \end{cases}$$

$$\mathbf{x} = \begin{bmatrix}x_1\\ x_2\end{bmatrix} = t\begin{bmatrix}1/3\\ 1\end{bmatrix}$$

Let the eigenvector $\mathbf{p_1} = \begin{bmatrix}1/3\\ 1\end{bmatrix}$.

(You could rescale this vector by a factor of 3, for example.)

Find an eigenvector for –1:

Solve the system $\left[(-1)I - A \middle| \mathbf{0}\right]$.

$$\left[\begin{array}{cc|c} -1-1 & 0-0 & 0 \\ 0-6 & -1-(-1) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -2 & 0 & 0 \\ -6 & 0 & 0 \end{array}\right] \quad \text{Now, } R_2 + (-3)R_1 \to R_2.$$

$$\left[\begin{array}{cc|c} -2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{Now, divide Row 1 through by (-2).}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 0$
Let $x_2 = t$.

$$\begin{cases} x_1 = 0 \\ x_2 = t \end{cases}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let the eigenvector $\mathbf{p_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

<u>Construct the diagonalizing matrix, *P*:</u>

$$\begin{aligned} P &= \begin{bmatrix} \mathbf{p_1} & \mathbf{p_2} \end{bmatrix} \\ &= \begin{bmatrix} 1/3 & 0 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

<u>Give *D*:</u>

Let $D = P^{-1}AP$.

$$\begin{aligned} D &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

(The eigenvalues are switched if the columns of *P* are switched.)

7) Prove that if A is an orthogonal matrix, then $|A|$ is either 1 or -1.
Show all steps! (5 points)

Proof 1

A is orthogonal

$\rightarrow A^{-1} = A^T$

$\rightarrow |A^{-1}| = \underbrace{|A^T|}_{=|A|}$

$\rightarrow \dfrac{1}{|A|} = |A| \quad (|A| \neq 0)$

$\rightarrow 1 = |A|^2$

$\rightarrow |A| = 1 \text{ or } -1$

Proof 2

A is orthogonal

$\rightarrow AA^T = I$

$\rightarrow |AA^T| = \underbrace{|I|}_{=1}$

$\rightarrow |A|\underbrace{|A^T|}_{=|A|} = 1$

$\rightarrow |A|^2 = 1$

$\rightarrow |A| = 1 \text{ or } -1$

TRUE or FALSE (12 points; 3 points each)

Assume that A is a real matrix in the statements below.

1) If the linear transformation $T: V \rightarrow W$ is an isomorphism, then $\text{Ker}(T)$ must only consist of a "zero" vector.

True False

If T is an isomorphism, then it is one-to-one, and only $\mathbf{0}_V$ gets mapped to $\mathbf{0}_W$ (i.e., Ker(T) = $\{\mathbf{0}_V\}$).

2) If A is a symmetric matrix, then A must be similar to a diagonal matrix.

True False

Any real symmetric matrix is diagonalizable.

3) If A is a square matrix that has 4 as an eigenvalue with algebraic multiplicity 3, then the eigenspace for $\lambda = 4$ must be three-dimensional.

True **False**

Absent additional information, we can only say that the 4-eigenspace can be 1-, 2-, or 3-dimensional.

4) If A is a square matrix whose columns form an orthogonal set of nonzero vectors, then A must be an orthogonal matrix.

True **False**

The column vectors may not all be unit vectors, which is what we require for an orthogonal matrix. The column vectors of A must form an orthonormal set of vectors for A to be an orthogonal matrix.