FINAL - SOLUTIONS

MATH 254 - SUMMER 2001 - KUNIYUKI

1) Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(8 points)

Idea: $\left[A \mid I\right] \to \left[I \mid A^{-1}\right]$.

$\left[A \mid I\right]$

$$\left[\begin{array}{ccc|ccc} 0 & 4 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$\frac{1}{4}R_2 \to R_2$

$\frac{1}{2}R_3 \to R_3$

So, $A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1/4 & 0 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$

2) Find the determinant of the matrix

$$A = \begin{vmatrix} 0 & 0 & 0 & \mathbf{2} \\ 3 & 1 & 2 & 7 \\ 6 & -2 & 3 & -1 \\ 5 & 0 & 0 & 4 \end{vmatrix}.$$

(10 points)

Let's expand along the first row, since it has the most zeros.
Its only nonzero entry is the "**2**"; the corresponding sign from the sign matrix is "-" [or you could observe that $(-1)^{i+j} = (-1)^{1+4} = (-1)^5 = -1$]. To get the submatrix for the corresponding minor, we delete the row and the column containing the "**2**".

$$\begin{vmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{2} \\ 3 & 1 & 2 & 7 \\ 6 & -2 & 3 & -1 \\ 5 & 0 & 0 & 4 \end{vmatrix} = \underbrace{(-1)}_{\text{from sign matrix}}(2)\begin{vmatrix} 3 & 1 & 2 \\ 6 & -2 & 3 \\ 5 & 0 & 0 \end{vmatrix} = -2\begin{vmatrix} 3 & 1 & 2 \\ 6 & -2 & 3 \\ \mathbf{5} & 0 & 0 \end{vmatrix}$$

Let's expand along the third row, since it has the most zeros.
Its only nonzero entry is the "**5**"; the corresponding sign from the sign matrix is "+" [or you could observe that $(-1)^{i+j} = (-1)^{3+1} = (-1)^4 = 1$]. To get the submatrix for the corresponding minor, we delete the row and the column containing the "**5**".

$$-2\begin{vmatrix} 3 & 1 & 2 \\ 6 & -2 & 3 \\ \mathbf{5} & \mathbf{0} & \mathbf{0} \end{vmatrix} = -2\left(5\begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix}\right)$$
$$= -2(5(3-(-4)))$$
$$= -2(5(7))$$
$$= -70$$

3) Let $\mathbf{v}_1 = (1,2)$ and $\mathbf{v}_2 = (3,5)$. Express (-7,-9) as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. There should be no unknowns in your final expression. (10 points)

Solve $\left[\begin{array}{cc|c} \mathbf{v}_1 & \mathbf{v}_2 & \begin{matrix} -7 \\ -9 \end{matrix} \end{array}\right]$ for c_1 and c_2, the weights for the linear combination.

$$\left[\begin{array}{cc|c} 1 & 3 & -7 \\ 2 & 5 & -9 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 3 & -7 \\ 0 & -1 & 5 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & -1 & 5 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & -5 \end{array}\right]$$

$R_2 - 2R_1 \to R_2$ $\qquad$ $R_1 + 3R_2 \to R_1$ $\qquad$ $-R_2 \to R_2$

So, $c_1 = 8$ and $c_2 = -5$.

$$\begin{bmatrix} -7 \\ -9 \end{bmatrix} = 8\mathbf{v}_1 - 5\mathbf{v}_2$$

4) The vectors **x**, **y**, and **z** are three vectors in R^n for some fixed n.
Prove that if **x** is orthogonal to both **y** and **z**, then **x** is orthogonal to any linear combination of **y** and **z**. (10 points)

Assume that **x** is orthogonal to **y** and **z**. Then, $\mathbf{x} \bullet \mathbf{y} = 0$, and $\mathbf{x} \bullet \mathbf{z} = 0$.
Let $c\mathbf{y} + d\mathbf{z}$ be any linear combination of **y** and **z**, where c and d are real numbers.
We need to show that $\mathbf{x} \bullet (c\mathbf{y} + d\mathbf{z}) = 0$.

$$\mathbf{x} \bullet (c\mathbf{y} + d\mathbf{z}) = (\mathbf{x} \bullet c\mathbf{y}) + (\mathbf{x} \bullet d\mathbf{z})$$
$$= c(\mathbf{x} \bullet \mathbf{y}) + d(\mathbf{x} \bullet \mathbf{z})$$
$$= c(0) + d(0)$$
$$= 0$$

5) The linear transformation $T: R^5 \to R^4$ is defined by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 0 & 4 & 3 & 0 \\ 0 & 1 & -2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 2 & 0 & 8 & 6 & 0 \end{bmatrix}.$$

Find a basis for the Range of T. (10 points)

Note that if we perform $R_4 - 2R_1 \to R_4$, we obtain the row-echelon "shape"

$$B = \begin{bmatrix} \mathbf{1} & 0 & 4 & 3 & 0 \\ 0 & \mathbf{1} & -2 & 5 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

You could also divide the third row by 3 to get reduced row-echelon (RRE) form.

Range(T) = Col(A), the column space of A. To get a basis for Col(A), take the columns of A that correspond to the pivot columns of the row-echelon "shape" B. In other words, we take columns 1, 2, and 5 of A.

Answer: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right\}$. (There are other possible answers.)

Note: You could also find a basis for $\text{Row}\left(A^T\right)$.

6) Orthogonally diagonalize the symmetric matrix

$$A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$$

by giving an orthogonal matrix P and a diagonal matrix D such that $D = P^{-1}AP$ (or, equivalently, $D = P^T AP$).

<u>Hint:</u> The eigenvalues of this matrix (A) are -3 and 5.

(25 points)

Solution:

Find two orthogonal eigenvectors of A:

Find an eigenvector for $\lambda_1 = -3$:

Solve the system $\left[-3I - A \middle| \mathbf{0}\right]$.

$$\left[\begin{array}{cc|c} -3-1 & 0-4 & 0 \\ 0-4 & -3-1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -4 & -4 & 0 \\ -4 & -4 & 0 \end{array}\right]$$

The reduced row-echelon (RRE) form of this matrix is:

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Now, parametrize the solution set:

$$x_1 + x_2 = 0 \rightarrow x_1 = -x_2$$
Let $x_2 = t$.

$$\begin{cases} x_1 = -t \\ x_2 = t \end{cases}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let the eigenvector $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

We need our eigenvector to be a unit vector, so let's normalize:

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Find an eigenvector for $\lambda_2 = 5$:

Solve the system $\left[5I - A \middle| \mathbf{0}\right]$.

$$\left[\begin{array}{cc|c} 5-1 & 0-4 & 0 \\ 0-4 & 5-1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 4 & -4 & 0 \\ -4 & 4 & 0 \end{array}\right]$$

The reduced row-echelon (RRE) form of this matrix is:

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Now, parametrize the solution set:

$$x_1 - x_2 = 0 \rightarrow x_1 = x_2$$

Let $x_2 = t$.

$$\begin{cases} x_1 = t \\ x_2 = t \end{cases}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let the eigenvector $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We need our eigenvector to be a unit vector, so let's normalize:

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

<u>Construct the diagonalizing orthogonal matrix, *P*:</u>

$$\begin{aligned} P &= \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \\ &= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \end{aligned}$$

<u>Give *D*:</u>

Let $D = P^{-1}AP$ or, equivalently, $P^{T}AP$.

$$\begin{aligned} D &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix} \end{aligned}$$

(The eigenvalues are switched if *P*' s columns are switched.)

7) A and P are $n \times n$ matrices. If det(A) = 4, and det(P) = 7, what is $\det\left(P^{-1}AP\right)$?
(6 points)

$$\begin{aligned} \left|P^{-1}AP\right| &= \left|P^{-1}\right||A||P| \\ &= \frac{1}{|P|}|A||P| \\ &= \frac{1}{7}(4)(7) \\ &= 4 \end{aligned}$$

Note that $\left|P^{-1}AP\right| = |A|$, regardless of what $|P|$ is, as long as it is nonzero and hence P is invertible. In other words, similar matrices have not just the same eigenvalues, but also the same determinant (in fact, the common determinant equals the product of eigenvalues, accounting for multiplicities).

TRUE OR FALSE

For each statement, circle "True" or "False" as appropriate.
You do not have to justify any of your responses.
(21 points; 3 points each)

(I noticed a bit too late that the True/False answers alternated.)

a) A homogeneous system of linear equations must be consistent.

True False

$A\mathbf{x} = \mathbf{0}$ always has $\mathbf{x} = \mathbf{0}$ as a solution.

b) If A, B, and C are $n \times n$ matrices, and if $AB = AC$, then it must be true that $B = C$.

True **False**

The statement is true if A is invertible, but it is not generally true.
For example, what if A is the zero matrix?

c) All elementary matrices are invertible.

True False

The inverse of an elementary matrix corresponds to the ERO that undoes the ERO represented by the elementary matrix.

d) The set of rational numbers is a subspace of **R**, the vector space of real numbers.

True **False**

The set of rational numbers is not closed under scalar multiplication. For example, 1 is a rational number, and π is a real scalar, but $\pi(1) = \pi$, which falls outside the set of rational numbers.

e) A set of five vectors in R^3 must be linearly dependent.

True False

Anytime the number of vectors exceeds the dimension of the space, the set of vectors must be linearly dependent.

f) A set of five vectors in R^3 must span R^3.

True **False**

For example, the five vectors could all be **0**.

g) If the linear transformation T: $R^3 \to R^4$ is defined by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} \mathbf{2} & 3 & 2 \\ 0 & \mathbf{1} & -5 \\ 0 & 0 & \mathbf{4} \\ 0 & 0 & 0 \end{bmatrix}$$, then T is a one-to-one transformation.

True False

Notice that A is already in row-echelon "shape" and that each column has a pivot position (i.e., the columns of A form a linearly independent set). T is then a one-to-one transformation. This is equivalent to saying that $A\mathbf{x} = \mathbf{b}$ can never have more than one solution.