QUIZ 1 - SOLUTIONS

MATH 254 - SUMMER 2001 - KUNIYUKI
CHAPTERS 1, 2, 3

1)

The given equations are already in standard form, and the like terms are lined up.

Now, write the corresponding augmented matrix and use row operations to obtain the reduced row-echelon (RRE) form.

$$\left[\begin{array}{cccc|c} \mathbf{3} & -2 & 6 & -5 & 2 \\ 0 & 1 & -3 & 0 & -5 \\ \mathbf{6} & -4 & 12 & -7 & 16 \end{array}\right]$$

We can use the "**3**" to turn the "**6**" into a "0".
Let's add (-2) times the first row to the third row.

$$R_3 + (-2)R_1 \rightarrow R_3$$

Old R_3	6	-4	12	-7		16
$(-2)R_1$	-6	4	-12	10		-4
New R_3	**0**	**0**	**0**	**3**		**12**

$$\left[\begin{array}{cccc|c} 3 & -2 & 6 & -5 & 2 \\ 0 & 1 & -3 & 0 & -5 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{3} & \mathbf{12} \end{array}\right]$$

Now, divide Row 3 through by 3 to change the "**3**" into a "1".

$$\frac{R_3}{3} \quad \text{or} \quad \frac{1}{3}R_3 \rightarrow R_3$$

$$\left[\begin{array}{cccc|c} 3 & -2 & 6 & \mathbf{-5} & 2 \\ 0 & 1 & -3 & 0 & -5 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{4} \end{array}\right]$$

This matrix is almost in row-echelon form (except for the fact that the "3" in the upper left corner should be a "1"). Still, we can "eliminate up" at this point.

Let's turn the "**-5**" into a "0" by adding 5 times Row 3 to Row 1.

$$R_1 + 5R_3 \rightarrow R_1$$

Old R_1	3	-2	6	-5		2
$5R_3$	0	0	0	5		20
New R_1	**3**	**-2**	**6**	**0**		**22**

$$\left[\begin{array}{cccc|c} \mathbf{3} & \mathbf{-2} & \mathbf{6} & \mathbf{0} & \mathbf{22} \\ 0 & 1 & -3 & 0 & -5 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Now, turn the "**-2**" into a "0" by adding 2 times Row 2 to Row 1.

$$R_1 + 2R_2 \to R_1$$

Old R_1	3	-2	6	0		22
$2R_2$	0	2	-6	0		-10
New R_1	**3**	**0**	**0**	**0**		**12**

$$\left[\begin{array}{cccc|c} \mathbf{3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{12} \\ 0 & 1 & -3 & 0 & -5 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Now, divide Row 1 through by 3 to change the "**3**" into a "1".

$$\frac{R_1}{3} \quad \text{or} \quad \frac{1}{3}R_1 \to R_1$$

$$\left[\begin{array}{cccc|c} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{4} \\ 0 & 1 & -3 & 0 & -5 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

We now have the reduced row-echelon (RRE) form of the original augmented matrix.

Notice that the third column in the coefficient matrix does <u>not</u> have a "leading 1", so x_3 is a free variable. The system is consistent, so there are infinitely many solutions.

Write the corresponding system.

$$\begin{cases} x_1 & & = 4 \\ & x_2 - 3x_3 & = -5 \\ & & x_4 = 4 \end{cases}$$

Move the free variable, x_3, to the right side in the second equation.

$$\begin{cases} x_1 = 4 \\ x_2 = 3x_3 - 5 \\ x_4 = 4 \end{cases}$$

Parametrization: Let $x_3 = t$.

Solution set in parametric form:

$$\begin{cases} x_1 = 4 \\ x_2 = 3t - 5 \\ x_3 = t \\ x_4 = 4 \end{cases}$$

t is any real number.

2)

$$X^T X = \begin{bmatrix} 1 & 1 & 1 \\ -5 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 2 & 50 \end{bmatrix}$$

Using the shortcut for 2×2 inverses:

$$\left(X^T X\right)^{-1} = \frac{1}{(3)(50) - (2)(2)} \begin{bmatrix} 50 & -2 \\ -2 & 3 \end{bmatrix}$$

$$= \frac{1}{146} \begin{bmatrix} 50 & -2 \\ -2 & 3 \end{bmatrix} \quad \leftarrow \text{I will accept this as your answer.}$$

$$= \begin{bmatrix} \frac{50}{146} & \frac{-2}{146} \\ \frac{-2}{146} & \frac{3}{146} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{25}{73} & -\frac{1}{73} \\ -\frac{1}{73} & \frac{3}{146} \end{bmatrix}$$

3)

If A is invertible, then A^{-1} exists.

$$A\mathbf{x} = \mathbf{b}$$
$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$
$$I\mathbf{x} = A^{-1}\mathbf{b}$$
$$\mathbf{x} = A^{-1}\mathbf{b}$$

<u>Uniqueness:</u> Assume that $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions.

$A\mathbf{x_1} = \mathbf{b}$ and $A\mathbf{x_2} = \mathbf{b}$

Then, $A\mathbf{x_1} = A\mathbf{x_2}$.

If A is invertible, then we can cancel the A on the left end of both sides.

We then conclude that $\mathbf{x_1} = \mathbf{x_2}$.

That is, any two solutions to the system must be the same.

4)

a)

Use EROs until we get an upper triangular matrix U.
Record the corresponding elementary matrices along the way.

$$A = \begin{bmatrix} 1 & 5 & -4 \\ 0 & 3 & 1 \\ 4 & 14 & -14 \end{bmatrix}$$

$R_3 + (-4)R_1 \to R_3$

$$\sim \begin{bmatrix} 1 & 5 & -4 \\ 0 & 3 & 1 \\ 0 & -6 & 2 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{-4} & 0 & 1 \end{bmatrix}$$

$R_3 + 2R_2 \to R_3$

$$\sim \underbrace{\begin{bmatrix} 1 & 5 & -4 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}}_{U} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mathbf{2} & 1 \end{bmatrix}$$

Now, construct L by taking I_3 and filling in the <u>opposites</u> of the boldfaced entries above.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix}$$

b)

Idea:

$$A\mathbf{x} = \mathbf{b}$$
$$L\underbrace{U\mathbf{x}}_{\mathbf{y}} = \mathbf{b}$$

First, solve $L\mathbf{y} = \mathbf{b}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ -28 \end{bmatrix}$$

Write the corresponding system:

$$\begin{cases} y_1 & = -10 \\ \quad y_2 & = \ \ 0 \\ 4y_1 - 2y_2 + y_3 & = -28 \end{cases}$$

Using forward substitution, we get:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 12 \end{bmatrix}$$

Second, solve $U\mathbf{x} = \mathbf{y}$.

$$\begin{bmatrix} 1 & 5 & -4 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 12 \end{bmatrix}$$

Write the corresponding system:

$$\begin{cases} x_1 + 5x_2 - 4x_3 & = -10 \\ \quad 3x_2 + x_3 & = \ \ 0 \\ \quad 4x_3 & = \ \ 12 \end{cases}$$

Using back substitution, we get our solution vector for the problem:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}$$

5)

Let's expand along the third column, since it has the most zeros.
Its only nonzero entry is the "**4**"; the corresponding sign from the sign matrix is "-" [or you could observe that $(-1)^{i+j} = (-1)^{4+3} = (-1)^7 = -1$]. To get the submatrix for the corresponding minor, we delete the row and the column containing the "**4**".

$$\begin{vmatrix} 1 & 7 & \mathbf{0} & -2 & 0 \\ 3 & -8 & \mathbf{0} & 5 & 5 \\ 4 & -3 & \mathbf{0} & 6 & 0 \\ \mathbf{-1} & \mathbf{7} & \mathbf{4} & \mathbf{-3} & \mathbf{15} \\ 4 & 2 & \mathbf{0} & 1 & 0 \end{vmatrix} = \underbrace{(-1)}_{\text{from sign matrix}} (4) \begin{vmatrix} 1 & 7 & -2 & \mathbf{0} \\ 3 & -8 & 5 & \mathbf{5} \\ 4 & -3 & 6 & \mathbf{0} \\ 4 & 2 & 1 & \mathbf{0} \end{vmatrix}$$

Let's expand along the fourth column, since it has the most zeros.

Its only nonzero entry is the "**5**"; the corresponding sign from the sign matrix is "+" [or you could observe that $(-1)^{i+j} = (-1)^{2+4} = (-1)^6 = 1$]. To get the submatrix for the corresponding minor, we delete the row and the column containing the "**5**".

$$= -4\begin{vmatrix} 1 & 7 & -2 & \mathbf{0} \\ \mathbf{3} & \mathbf{-8} & \mathbf{5} & \mathbf{5} \\ 4 & -3 & 6 & \mathbf{0} \\ 4 & 2 & 1 & \mathbf{0} \end{vmatrix}$$

$$= -4\left(5\begin{vmatrix} 1 & 7 & -2 \\ 4 & -3 & 6 \\ 4 & 2 & 1 \end{vmatrix}\right)$$

$$= -20\begin{vmatrix} 1 & 7 & -2 \\ 4 & -3 & 6 \\ 4 & 2 & 1 \end{vmatrix}$$

You could then use Sarrus's Rule or expansion by minors/cofactors to compute the "3×3" determinant, which turns out to be 85.

$$= -20(85)$$
$$= -1700$$

6)

A has order *n* = 10. See Theorem 3.6 on p.130.

$$\begin{aligned} |2A| &= 2^n|A| \\ &= 2^{10}(3) \\ &= (1024)(3) \\ &= 3072 \end{aligned}$$

7)

If *A* is invertible, then A^{-1} exists.

$$AA^{-1} = I$$
$$|AA^{-1}| = |I|$$
$$\underbrace{|A|}_{\neq 0}|A^{-1}| = 1$$
$$|A^{-1}| = \frac{1}{|A|}$$

TRUE or FALSE

1) **False**. The corresponding coefficient matrix is "fat" - if the system is consistent, then there are one or more free variables that ensure the existence of infinitely many solutions.

2) **True**. For example, consider the system

$$\begin{cases} x = 0 \\ y = 0 \\ 2x = 0 \end{cases}$$

Augmented matrix: $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $(x = 0, y = 0)$ is then the unique solution.

3) **True**. The 7 "leading ones" must lie along the main diagonal. In RRE form, it is required that the other entries in any column containing a "leading one" must be zeros. I_7 is the only 7×7 matrix that satisfies these conditions.

4) **False**. It is true that:

$$\begin{aligned} (A-B)^2 &= (A-B)(A-B) \\ &= A(A-B) - B(A-B) \\ &= AA - AB - BA + BB \\ &= A^2 - AB - BA + B^2 \end{aligned}$$

However, matrix multiplication is not commutative. There are matrices A and B for which $AB \neq BA$; then, the last expression does not equal $A^2 - 2AB + B^2$.