QUIZ 2 - SOLUTIONS

MATH 254 - SUMMER 2001 - KUNIYUKI
CHAPTER 4

1)

$$3\mathbf{v} - 4\mathbf{w} = 3\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} - 4\begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ -6 \end{bmatrix} - \begin{bmatrix} -16 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 9-(-16) \\ 0-8 \\ -6-4 \end{bmatrix} = \begin{bmatrix} 25 \\ -8 \\ -10 \end{bmatrix}$$

2)

(Deleted)

3)

The set (say, "W") is a nonempty subset of R^5; $\mathbf{0}$ is guaranteed to be in it.

Let $\mathbf{x_1}$ and $\mathbf{x_2}$ be any two members of W. Then,

$A\mathbf{x_1} = \mathbf{0}$, and $A\mathbf{x_2} = \mathbf{0}$.

Prove closure of *W* under vector addition:

Show $A(\mathbf{x_1} + \mathbf{x_2}) = \mathbf{0}$.

$$\begin{aligned} A(\mathbf{x_1} + \mathbf{x_2}) &= A\mathbf{x_1} + A\mathbf{x_2} \\ &= \mathbf{0} + \mathbf{0} \\ &= \mathbf{0} \end{aligned}$$

So, $\mathbf{x_1} + \mathbf{x_2}$ is in W.

Prove closure of *W* under scalar multiplication:

Let c be any real scalar.

Show $A(c\mathbf{x_1}) = \mathbf{0}$.

$$\begin{aligned} A(c\mathbf{x_1}) &= c(A\mathbf{x_1}) \\ &= c(\mathbf{0}) \\ &= \mathbf{0} \end{aligned}$$

So, $c\mathbf{x_1}$ is in W.

Note: You could prove both types of closure simultaneously by showing $A(\mathbf{x_1} + c\mathbf{x_2}) = \mathbf{0}$.

Therefore, W is a subspace of R^5.

4)

a) The set of all polynomials in x whose degree is exactly three

Yes **No**

Note: The answer would have been "Yes" if the set were
"The set of all polynomials in x with degree three or less, including 0."

b) The set of all real 4×3 matrices

Yes No

c) The set of standard basis vectors in R^5

Yes **No**

For example, **0** is not a standard basis vector.

d) The set of all real 2×2 matrices in upper triangular form

Yes No

This is the set $\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| \, a, b, c \text{ are real numbers} \right\}$.

e) The set $\{(x, 0, 2x) : x \text{ is a real number}\}$

Yes No

5)

a) {(4, 3), (0, 0)}

Linearly independent **Linearly dependent**

Any set with **0** is linearly dependent.

b) {(2, 5, -4), (4, 10, -8)}

Linearly independent **Linearly dependent**

Any set with two vectors is linearly dependent if and only if one is a multiple of the other.

c) {(1, 0, 2), (2, 4, 4), (0, 1, 0)}

Linearly independent **Linearly dependent**

Consider the matrix with the given vectors as its columns:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 1 \\ 2 & 4 & 0 \end{bmatrix} \underset{R_3 - 2R_1 \to R_3}{\sim} \begin{bmatrix} \mathbf{1} & 2 & 0 \\ 0 & \mathbf{4} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Not every column in the row-echelon "shape" has a **pivot position**, so the columns of the original matrix form a linearly dependent set.

d) {(4, 0, 0), (2, 4, 0), (1, 3, -2)}

Linearly independent Linearly dependent

Consider the matrix with the given vectors as its columns:

$$\begin{bmatrix} \mathbf{4} & 2 & 1 \\ 0 & \mathbf{4} & 3 \\ 0 & 0 & \mathbf{-2} \end{bmatrix}$$

Every column in this row-echelon "shape" has a **pivot position**, so the columns form a linearly independent set.

e) Any set of 10 vectors in R^7.

Linearly independent **Linearly dependent**

If the number of vectors in the set (10) exceeds the dimension of the space (7), the set must be linearly dependent.

f) A set of three vectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ in R^4 with the property that $\mathbf{v_1} - 3\mathbf{v_2} + 2\mathbf{v_3} = \mathbf{0}$.

Linearly independent **Linearly dependent**

You are given that $c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} = \mathbf{0}$ has a nontrivial solution.

6)

a)

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

b)

There are 4 vectors in the standard basis, so the dimension of the space is **4**.

7)

True, because a set of n vectors in an n-dimensional space has the following property: Either the set is both linearly independent and a spanning set for the space (i.e., it's a basis), or it is neither linearly independent nor a spanning set.

8)

a)

Rank = **3**, because the row-echelon "shape" B has 3 pivot positions:

$$B = \begin{bmatrix} \mathbf{1} & -3 & 0 & 5 & 0 \\ 0 & 0 & \mathbf{1} & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b)

To get a basis for Row(A), take the pivot (nonzero) rows of the row-echelon "shape" B.

$$\left\{(1, -3, 0, 5, 0),\ \left(0, 0, 1, -\frac{3}{2}, 0\right),\ (0, 0, 0, 0, 1)\right\}$$

c)

To get a basis for Col(A), take the columns of A that correspond to the pivot columns of the row-echelon "shape" B. In other words, we take columns 1, 3, and 5 of A.

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix} \right\}$$

d)

The system $A\mathbf{x} = \mathbf{0}$ is equivalent to the system $B\mathbf{x} = \mathbf{0}$.

$$[A \mid \mathbf{0}] \sim [B \mid \mathbf{0}]$$

$$\left[\begin{array}{ccccc|c} 1 & -3 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Write the corresponding system:

$$\begin{cases} x_1 - 3x_2 \qquad + 5x_4 \qquad = 0 \\ \qquad x_3 - \frac{3}{2}x_4 \qquad = 0 \\ \qquad\qquad x_5 = 0 \\ \qquad\qquad 0 = 0 \quad \leftarrow \text{can omit} \end{cases}$$

The nonpivot columns of *B* correspond to the free variables: x_2 and x_4.

Move the free variables in the equations over to the right side.

$$\begin{cases} x_1 = 3x_2 - 5x_4 \\ x_3 = \qquad \frac{3}{2}x_4 \\ x_5 = 0 \end{cases}$$

Let

$$x_2 = t$$
$$x_4 = u$$

$$\begin{cases} x_1 = 3t - 5u \\ x_2 = t \\ x_3 = \qquad \frac{3}{2}u \\ x_4 = \qquad u \\ x_5 = 0 \end{cases}$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = t\underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v_1}} + u\underbrace{\begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v_2}}$, *t* and *u* are any real numbers

A basis for *N*(*A*) is $\{\mathbf{v_1}, \mathbf{v_2}\}$.

9) For any matrix *A*,

$$\text{rank}(A) + \text{nullity}(A) = \underbrace{n}_{\substack{\text{the number of} \\ \text{columns in } A}}$$

Here, $4 + \text{nullity}(A) = 10$

$$\text{nullity}(A) = \mathbf{6}$$