QUIZ 3 - SOLUTIONS

MATH 254 - SUMMER 2001 - KUNIYUKI
CHAPTER 5, 6

1)

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{(4)^2 + (2)^2 + (-3)^2}}(4, 2, -3) = \frac{1}{\sqrt{29}}(4, 2, -3)$$

$$= \left(\frac{4}{\sqrt{29}}, \frac{2}{\sqrt{29}}, \frac{-3}{\sqrt{29}}\right)$$

2)

$$\cos\theta = \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} = \frac{(3, 1, 0) \bullet (2, -3, 5)}{\sqrt{(3)^2 + (1)^2 + (0)^2}\,\sqrt{(2)^2 + (-3)^2 + (5)^2}} = \frac{6 - 3 + 0}{\sqrt{10}\,\sqrt{38}} = \frac{3}{\sqrt{380}} \approx 0.153897$$

$$\theta = \cos^{-1}\left(\frac{3}{\sqrt{380}}\right)$$

$$\approx 81.15^\circ \quad \text{or} \quad 1.416 \text{ radians}$$

3)

If $\mathbf{v}$ is orthogonal to $\mathbf{w}$, then their dot product is 0.

4)

Let $\mathbf{w_1} = \mathbf{v_1} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$.

Let

$$\mathbf{w_2} = \mathbf{v_2} - \underbrace{\text{proj}_{\mathbf{w_1}}\mathbf{v_2}}_{\left(\frac{\mathbf{v_2} \bullet \mathbf{w_1}}{\mathbf{w_1} \bullet \mathbf{w_1}}\right)\mathbf{w_1}}$$

$$= \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix} - \frac{(-3, 14, -7) \bullet (3, -4, 5)}{(3, -4, 5) \bullet (3, -4, 5)}\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix} - \frac{-100}{50}\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -3+2(3) \\ 14+2(-4) \\ -7+2(5) \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$$

Finally, we must normalize the vectors in $\{\mathbf{w_1}, \mathbf{w_2}\}$ to get the desired orthonormal basis $\{\mathbf{u_1}, \mathbf{u_2}\}$.

$$\mathbf{u_1} = \frac{1}{\|\mathbf{w_1}\|}\mathbf{w_1} = \frac{1}{\sqrt{(3)^2+(-4)^2+(5)^2}}\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{50}}\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{50} \\ -4/\sqrt{50} \\ 5/\sqrt{50} \end{bmatrix}$$

$$\mathbf{u_2} = \frac{1}{\|\mathbf{w_2}\|}\mathbf{w_2} = \frac{1}{\sqrt{(3)^2+(6)^2+(3)^2}}\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{54}}\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{54} \\ 6/\sqrt{54} \\ 3/\sqrt{54} \end{bmatrix}$$

5)

T is a linear transformation, so

$$T(2\mathbf{b_1}+5\mathbf{b_2}) = 2T(\mathbf{b_1})+5T(\mathbf{b_2})$$

$$= 2\begin{bmatrix} 3 \\ 5 \end{bmatrix} + 5\begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 10 \end{bmatrix} + \begin{bmatrix} -5 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 30 \end{bmatrix}$$

6)

a) A is 4×5.

b) The dimension of Ker(T) is nullity(T).

$$\text{rank}(T)+\text{nullity}(T) = n$$
$$3+\text{nullity}(T) = 5$$
$$\text{nullity}(T) = 2$$

Answer: 2

7)

a)

The dimension of Range(T) is rank(T), which equals the number of pivot positions in a row-echelon shape of A (which A is already in). Answer: 4.

$$A = \begin{bmatrix} \mathbf{1} & -2 & 4 & 5 & 0 \\ 0 & 0 & \mathbf{3} & -1 & 0 \\ 0 & 0 & 0 & \mathbf{2} & 6 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

b)

nullity(T) = 1, since there is only one nonpivot column. Also:

$$\begin{aligned} \text{rank}(T) + \text{nullity}(T) &= n \\ 4 + \text{nullity}(T) &= 5 \\ \text{nullity}(T) &= 1 \end{aligned}$$

c)

Yes, T is onto, because each row has a pivot position.
Note: This means that Range(T) = Col(A) = the codomain, R^4.

8)

If A is similar to B, then there exists an invertible matrix P such that $A = P^{-1}BP$.

$$\begin{aligned} A^2 &= AA \\ &= \left(P^{-1}BP\right)\left(P^{-1}BP\right) \quad \text{for some invertible matrix } P \\ &= P^{-1}B\underbrace{PP^{-1}}_{I}BP \\ &= P^{-1}BIBP \\ &= P^{-1}BBP \\ &= P^{-1}B^2P \end{aligned}$$

Therefore, A^2 is similar to B^2.

9)

a) Let's call the standard matrix "A".

<u>Method 1</u>

$T(1,0) = (1,2)$
$T(0,1) = (3,-1)$

$$A = \begin{bmatrix} T(1,0) & T(0,1) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

<u>Method 2</u>

List the components of $T(x,y)$ in rows and line up like terms:

$$\begin{aligned} x + 3y \\ 2x - y \end{aligned}$$

The coefficients then make up A:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

b)

The transition matrix from B' to B is:

$$P = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

The transition matrix from B to B' is:

$$\begin{aligned} P^{-1} &= \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}^{-1} \\ &= \frac{1}{\det(P)} \begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix} \\ &= \frac{1}{-5} \begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix} \\ &= -\frac{1}{5} \begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix} \end{aligned}$$

So, the matrix for T relative to B' is:

$$\begin{aligned} A' &= P^{-1}AP \\ &= -\frac{1}{5} \begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \\ &= -\frac{1}{5} \begin{bmatrix} 43 & 31 \\ -54 & -43 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{43}{5} & -\frac{31}{5} \\ \frac{54}{5} & \frac{43}{5} \end{bmatrix} \end{aligned}$$