QUIZ 4 - SOLUTIONS

MATH 254 - SUMMER 2001 - KUNIYUKI
CHAPTER 7, 8

1) Find the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. (20 points)

Find the eigenvalues:

Solve $|\lambda I - A| = 0$.

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} \lambda - 1 & 0 - 1 \\ 0 - (-2) & \lambda - 4 \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 4 \end{vmatrix} = 0$$

$$(\lambda - 1)(\lambda - 4) - (-1)(2) = 0$$

$$\lambda^2 - 5\lambda + 4 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 3)(\lambda - 2) = 0$$

$$\lambda = 3 \text{ or } \lambda = 2$$

2) Diagonalize the matrix $A = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$ by giving matrices P and D such that $D = P^{-1}AP$, where D is diagonal. You do not have to give P^{-1}. Hint: The eigenvalues of this matrix are 3 and 4. (30 points)

Find two linearly independent eigenvectors of A:

Since we have $n = 2$ distinct real eigenvalues, A is guaranteed to be diagonalizable.

Find an eigenvector for $\lambda_1 = 3$:

Solve the system $\left[3I - A \,|\, \mathbf{0}\right]$.

$$\left[\begin{array}{cc|c} 3-2 & 0-2 & 0 \\ 0-(-1) & 3-5 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$$R_2 - R_1 \to R_2$$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 - 2x_2 = 0 \to x_1 = 2x_2$
Let $x_2 = t$.

$$\begin{cases} x_1 = 2t \\ x_2 = t \end{cases}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let the eigenvector $\mathbf{p_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Find an eigenvector for $\lambda_2 = 4$:

Solve the system $\left[4I - A \middle| \mathbf{0}\right]$.

$$\left[\begin{array}{cc|c} 4-2 & 0-2 & 0 \\ 0-(-1) & 4-5 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 - \frac{1}{2}R_1 \to R_2$$

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\frac{1}{2}R_1 \to R_1$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 - x_2 = 0 \to x_1 = x_2$
Let $x_2 = t$.

$$\begin{cases} x_1 = t \\ x_2 = t \end{cases}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let the eigenvector $\mathbf{p_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

<u>Construct the diagonalizing matrix, *P*:</u>

$$\begin{aligned} P &= \begin{bmatrix} \mathbf{p_1} & \mathbf{p_2} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

<u>Give *D*:</u>

Let $D = P^{-1}AP$.

$$\begin{aligned} D &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{(The eigenvalues are switched if } P\text{'s columns are switched.)} \end{aligned}$$

3) The matrix *A* below is an orthogonal matrix. Write A^{-1}, the inverse matrix. (5 points)

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$$

Since *A* is an orthogonal matrix, $A^{-1} = A^T$, which is:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

4) Let $A = \begin{bmatrix} 2i & 3-i \\ 4 & 5i \end{bmatrix}$ and $B = \begin{bmatrix} 2+i & -i \\ 4i & 3 \end{bmatrix}$. Find the matrix product AB. The entries of your final answer must be in standard form. (15 points)

$$\begin{aligned}
AB &= \begin{bmatrix} 2i & 3-i \\ 4 & 5i \end{bmatrix}\begin{bmatrix} 2+i & -i \\ 4i & 3 \end{bmatrix} \\
&= \begin{bmatrix} (2i)(2+i)+(3-i)(4i) & (2i)(-i)+(3-i)(3) \\ (4)(2+i)+(5i)(4i) & (4)(-i)+(5i)(3) \end{bmatrix} \\
&= \begin{bmatrix} 4i+2i^2+12i-4i^2 & -2i^2+9-3i \\ 8+4i+20i^2 & -4i+15i \end{bmatrix} \\
&= \begin{bmatrix} 16i-2\underbrace{i^2}_{(-1)} & -2\underbrace{i^2}_{(-1)}+9-3i \\ 8+4i+20\underbrace{i^2}_{(-1)} & 11i \end{bmatrix} \\
&= \begin{bmatrix} 16i+2 & 2+9-3i \\ 8+4i-20 & 11i \end{bmatrix} \\
&= \begin{bmatrix} 2+16i & 11-3i \\ -12+4i & 11i \end{bmatrix}
\end{aligned}$$

5) Let $z = 4 - 5i$. Find $|z|$. (5 points)

$$\begin{aligned}
|z| &= |4-5i| \\
&= \sqrt{(4)^2+(-5)^2} \\
&= \sqrt{16+25} \\
&= \sqrt{41}
\end{aligned}$$

6) Perform the division and write your final answer in standard form:

$$\frac{3i}{2-5i}$$

(10 points)

$$\begin{aligned}
\frac{3i}{2-5i} &= \frac{3i}{2-5i} \bullet \frac{2+5i}{2+5i} \quad (2+5i \text{ is the complex conjugate of the denominator}) \\
&= \frac{3i(2+5i)}{(2-5i)(2+5i)} \\
&= \frac{6i+15\overbrace{i^2}^{(-1)}}{(2)^2+(5)^2} \quad \left(\text{Use: } (a-bi)(a+bi)=a^2+b^2\right) \\
&= \frac{-15+6i}{29} \\
&= -\frac{15}{29}+\frac{6}{29}i
\end{aligned}$$

7) The linear transformation $T: C^2 \to C^2$ is given by $T(\mathbf{v}) = A\mathbf{v}$ where

$$A = \begin{bmatrix} 0 & 4 \\ 4i & 2i \end{bmatrix}$$

(15 points total)

a) Find the image of $\mathbf{v} = \begin{bmatrix} 4 \\ 3+2i \end{bmatrix}$ under this transformation. (6 points)

$$\begin{aligned}
T(\mathbf{v}) &= A\mathbf{v} \\
&= \begin{bmatrix} 0 & 4 \\ 4i & 2i \end{bmatrix} \begin{bmatrix} 4 \\ 3+2i \end{bmatrix} \\
&= \begin{bmatrix} (0)(4)+(4)(3+2i) \\ (4i)(4)+(2i)(3+2i) \end{bmatrix} \\
&= \begin{bmatrix} 0+12+8i \\ 16i+6i+4i^2 \end{bmatrix} \\
&= \begin{bmatrix} 12+8i \\ 22i+4(-1) \end{bmatrix} \\
&= \begin{bmatrix} 12+8i \\ -4+22i \end{bmatrix}
\end{aligned}$$

b) Find the preimage of $\mathbf{w} = \begin{bmatrix} 12 \\ 8i \end{bmatrix}$. (9 points)

Solve $\left[A \middle| \mathbf{w} \right]$.

$$\left[\begin{array}{cc|c} 0 & 4 & 12 \\ 4i & 2i & 8i \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{cc|c} 4i & 2i & 8i \\ 0 & 4 & 12 \end{array}\right]$$

$\frac{1}{4i} R_1 \to R_1$

$$\left[\begin{array}{cc|c} 1 & 1/2 & 2 \\ 0 & 4 & 12 \end{array}\right]$$

$\frac{1}{4} R_2 \to R_2$

$$\left[\begin{array}{cc|c} 1 & 1/2 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

$R_1 - \frac{1}{2} R_2 \to R_1$

$$\left[\begin{array}{cc|c} 1 & 0 & 2-(3/2) \\ 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 3 \end{array}\right]$$

Answer: $\left\{ \begin{bmatrix} 1/2 \\ 3 \end{bmatrix} \right\}$.