

SECTION 4.7: INVERSE TRIG FUNCTIONS

You may want to review [Section 1.8](#) on inverse functions.

PART A : GRAPH OF $\sin^{-1} x$ (or $\arcsin x$)

Warning: Remember that f^{-1} denotes function inverse, not multiplicative inverse (or reciprocal). Usually, $f^{-1} \neq \frac{1}{f}$. In particular, $\sin^{-1} x \neq \frac{1}{\sin x}$, or $\csc x$. We **can** say that $(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$. Although it is often helpful in Calculus to rewrite $\sin^n x$ as $(\sin x)^n$, this is **not** true of $\sin^{-1} x$, because -1 is **not** an exponent in that case. However, -1 **does** act as an exponent in $(\sin x)^{-1}$.

If $f(x) = \sin x$, and the domain is \mathbf{R} (which is, after all, the implied domain), then f is **not** a one-to-one function, and it has no inverse **function**.

We want to define an inverse sine (or “arcsine”) function $f^{-1}(x) = \sin^{-1} x$ (or $\arcsin x$). To do so, we must restrict the domain of $f(x) = \sin x$ so that it is a one-to-one function whose graph passes the HLT (Horizontal Line Test).

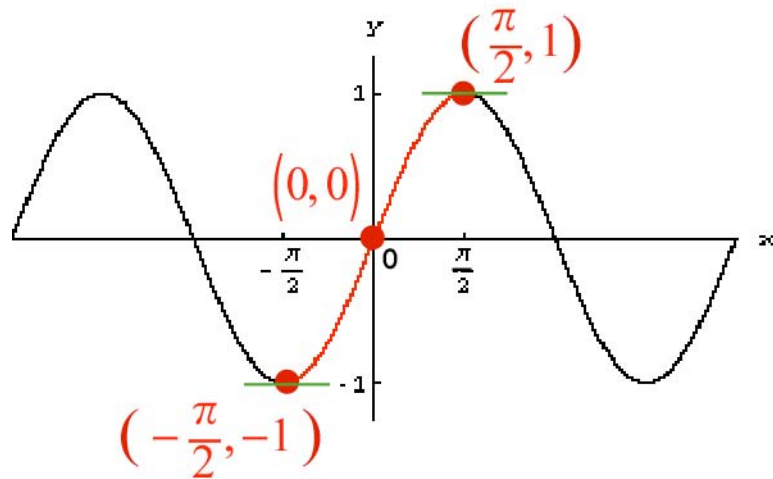
What should this restricted domain be? It should be an x -interval on which the $\sin x$ graph:

- 1) Passes the HLT, and
- 2) Is as “tall” as the original, unrestricted $\sin x$ graph. In other words, we would like the range to be the same as before.

It is universally agreed that we take the x -interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ as our restricted domain.

The resulting range for our $\sin x$ function remains $[-1, 1]$.

The resulting graph is in red below:
 (The x - and y -axes are scaled differently.)

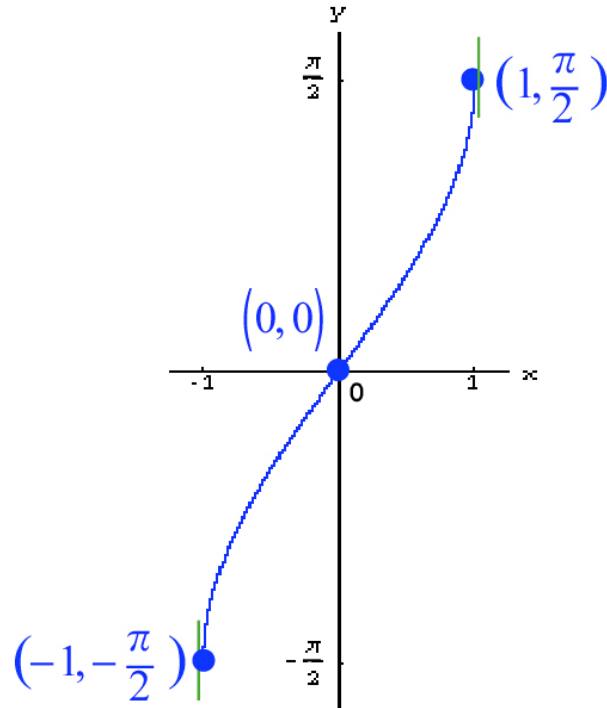


Observe that:

- The function increases on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- The graph switches from concave up to concave down at $(0, 0)$.

It may be easier to remember that the graph is a snake of finite length that has horizontal (one-sided) tangent lines (in green) at its endpoints.

The graph of $f^{-1}(x) = \sin^{-1} x$ (or $\arcsin x$), the arcsine function, is obtained by switching the x - and y -coordinates of all the points on the red graph we just saw. (Reflecting the red graph about the line $y = x$ may be hard to visualize.) We obtain:



Observe that:

- The inverse function also increases, but on the interval $[-1, 1]$.
The three indicated points above suggest this.

- However, the graph switches from concave down to concave up at $(0, 0)$.
It may be easier to remember that the graph is a snake of finite length that has vertical (one-sided) tangent lines (in green) at its endpoints.

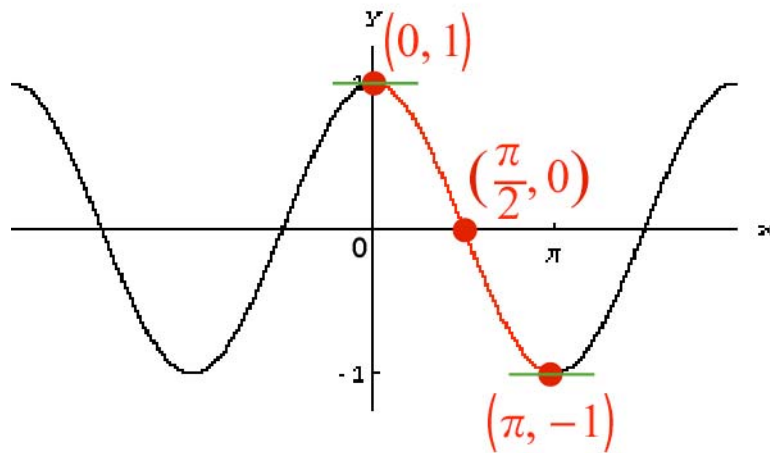
Remember that, for a pair of inverse functions, the domain of one is the range of the other.

	Domain	Range
$\sin x$ (restricted)	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[-1, 1]$
$\sin^{-1} x$ (or $\arcsin x$)	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

PART B: GRAPH OF $\cos^{-1} x$ (or $\arccos x$)

It is universally agreed that we take the x -interval $[0, \pi]$ as our restricted domain for $f(x) = \cos x$. The resulting range remains $[-1, 1]$.

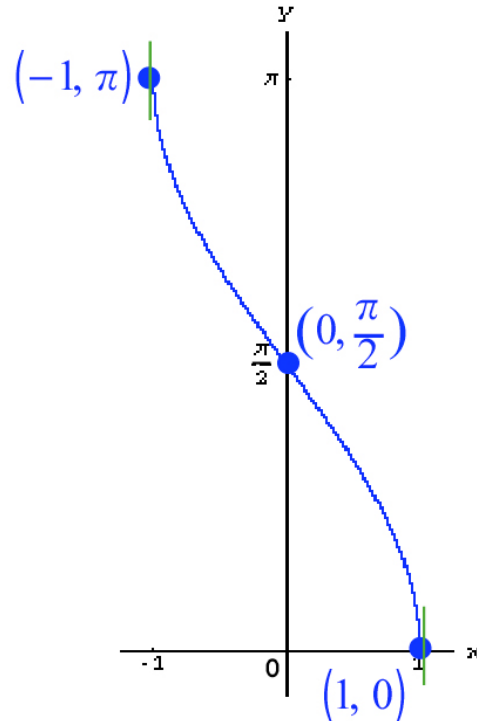
The resulting graph is in red below:
(The x - and y -axes are scaled differently.)



Observe that:

- The function decreases on the interval $[0, \pi]$.
- The graph switches from concave down to concave up at the “midpoint” $(\frac{\pi}{2}, 0)$. It may be easier to remember that, just as for $\sin x$, the graph is a snake of finite length that has horizontal (one-sided) tangent lines (in green) at its endpoints.

The graph of $f^{-1}(x) = \cos^{-1} x$ (or $\arccos x$), the arccosine function, is obtained by switching the x - and y -coordinates of all the points on the red graph we just saw. We obtain:



Observe that:

- The inverse function also decreases, but on the interval $[-1, 1]$.
The three indicated points above suggest this.
- However, the graph switches from concave up to concave down at the “midpoint” $\left(0, \frac{\pi}{2}\right)$. It may be easier to remember that, just as for $\sin^{-1} x$, the graph is a snake of finite length that has vertical (one-sided) tangent lines (in green) at its endpoints.

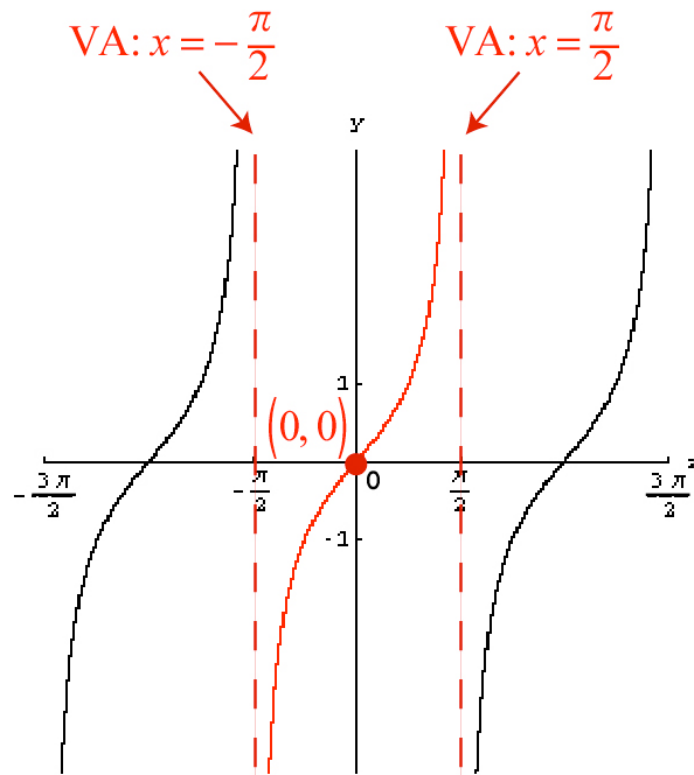
Remember that, for a pair of inverse functions, the domain of one is the range of the other.

	Domain	Range
$\cos x$ (restricted)	$[0, \pi]$	$[-1, 1]$
$\cos^{-1} x$ (or $\arccos x$)	$[-1, 1]$	$[0, \pi]$

PART C : GRAPH OF $\tan^{-1} x$ (or arctan x)

It is universally agreed that we take the **open** x -interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ as our restricted domain for $f(x) = \tan x$. The resulting range remains \mathbf{R} , or $(-\infty, \infty)$.

The resulting graph is in red below:
(The x - and y -axes are scaled differently.)



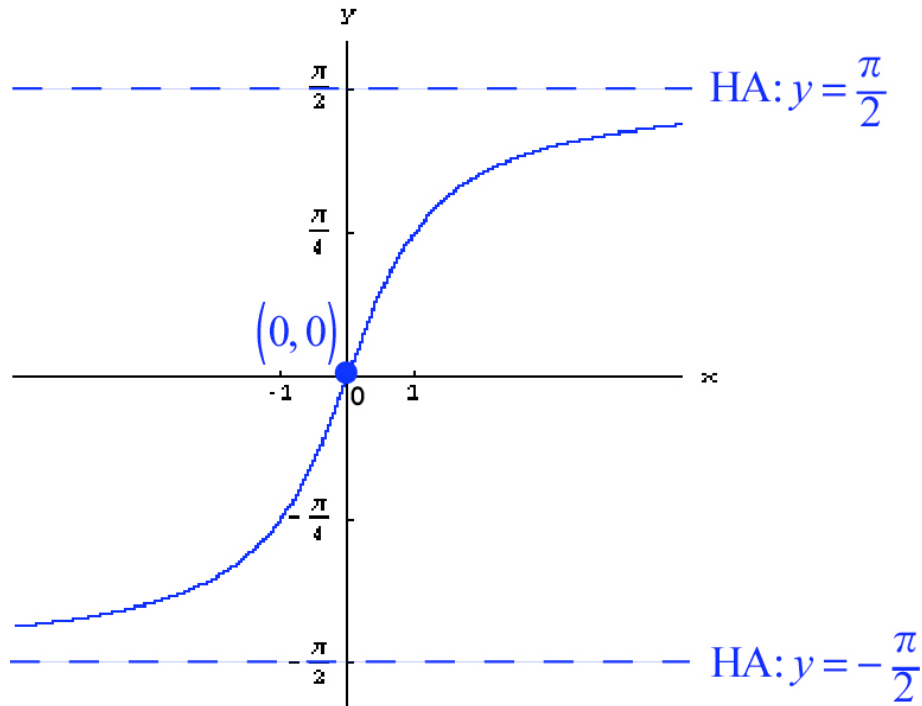
Observe that:

- The function increases on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- The graph switches from concave down to concave up at $(0, 0)$.

It may be easier to remember that the graph is a snake of **infinite** length (unlike before) that approaches the **vertical** asymptotes (VAs)

at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

The graph of $f^{-1}(x) = \tan^{-1} x$ (or $\arctan x$), the arctangent function, is obtained by switching the x - and y -coordinates of all the points on the red graph we just saw. We obtain:



Observe that:

- The inverse function also increases, but on all of \mathbf{R} .
- However, the graph switches from concave up to concave down at $(0,0)$.

It may be easier to remember that the graph is a snake of **infinite**

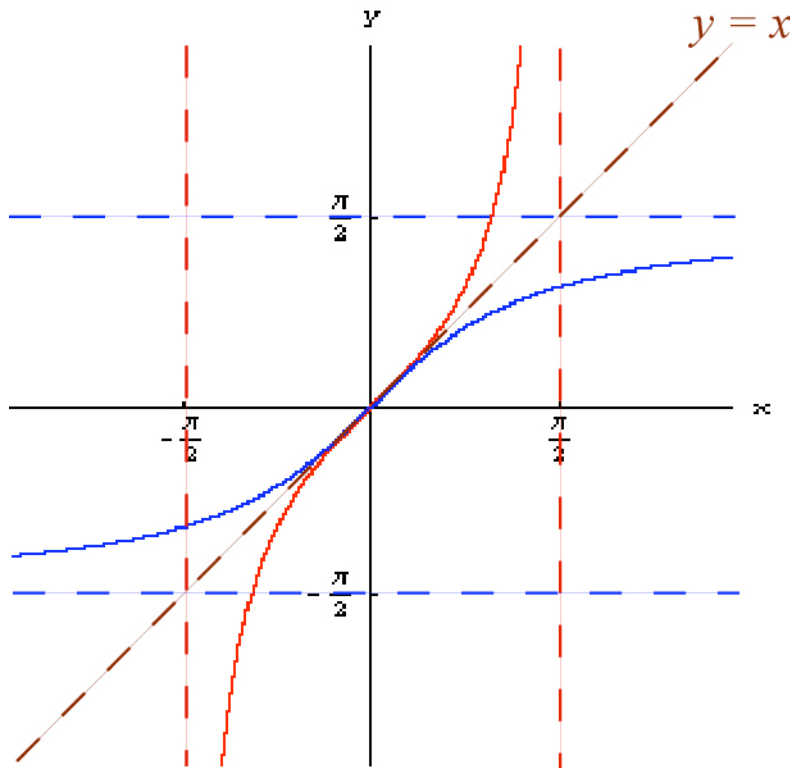
length that approaches the **horizontal** asymptotes (HAs) at $y = -\frac{\pi}{2}$

and $y = \frac{\pi}{2}$.

Remember that, for a pair of inverse functions, the domain of one is the range of the other.

	Domain	Range
$\tan x$ (restricted)	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	\mathbf{R} , or $(-\infty, \infty)$
$\tan^{-1} x$ (or $\arctan x$)	\mathbf{R} , or $(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

You may have been able to visualize reflecting the red graph about the line $y = x$ (in brown) in order to obtain the blue graph.



Warning: The line $y = x$ may appear too flat or too steep if the x - and y -axes are scaled differently.

PART D: OTHER INVERSE TRIG FUNCTIONS

The problem with the \csc^{-1} , \sec^{-1} , and \cot^{-1} functions is that their ranges are not universally agreed upon! (In other words, there is no universal agreement about how the domains of the \csc , \sec , and \cot functions should be restricted.) Different ranges may be used for different purposes.

For example, the range of \sec^{-1} may include angles from Quadrant II (as for \cos^{-1}) or from Quadrant III (which tends to be more convenient in Calculus, because \tan is positive there).

PART E: REMEMBERING THE RANGES OF INVERSE TRIG FUNCTIONS

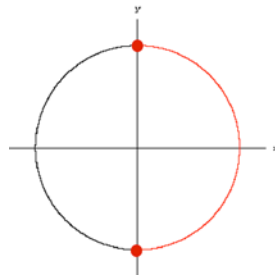
Here are some tricks:

\sin^{-1}

Remember that the range of \sin^{-1} is the restricted domain of \sin .

We could recall the red graph in [Notes 4.73](#) and find the set of x -coordinates picked up by the graph.

We may also recall the Unit Circle. We want to focus on an arc of the Unit Circle that “picks up” **all** of the possible \sin values (corresponding to y -coordinates on the circle) **exactly once** (so that we force the restricted \sin function to be one-to-one). We pick the right semicircle, as opposed to the left one.

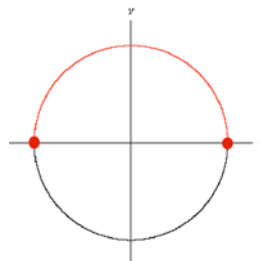


The interval of angles we typically associate with this arc is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Note: We may say “angles” when “angle measures” may be more appropriate.

\cos^{-1}

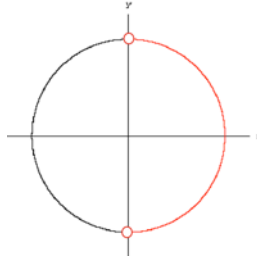
Similarly, we could recall the red graph in [Notes 4.75](#), or we could focus on an arc of the Unit Circle that “picks up” **all** of the possible \cos values (corresponding to x -coordinates on the circle) **exactly once**. We pick the top semicircle, as opposed to the bottom one.



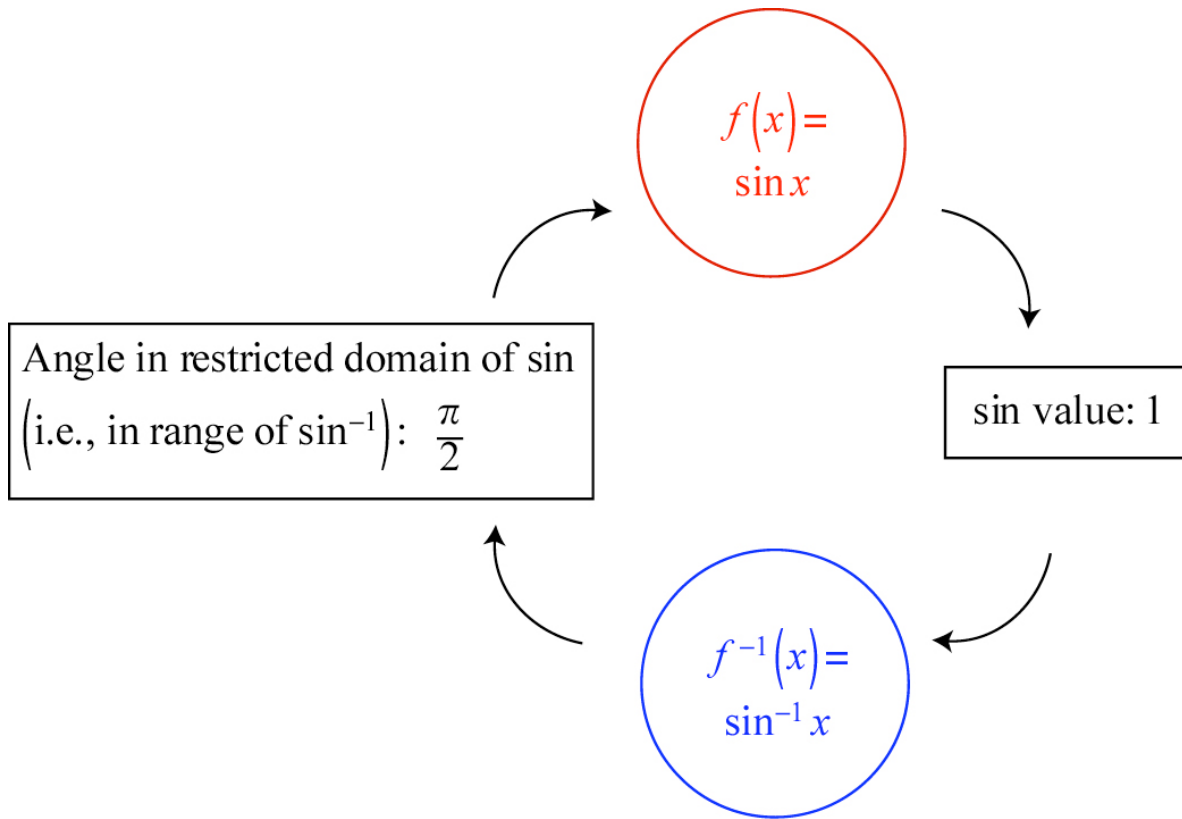
The interval of angles we typically associate with this arc is $[0, \pi]$.

\tan^{-1}

Similarly, we could recall the red graph in [Notes 4.77](#), or we could focus on an arc of the Unit Circle that “picks up” **all** of the possible \tan values (corresponding to slopes of terminal sides of standard angles) **exactly once**. As for \sin^{-1} , we pick the right semicircle, as opposed to the left one, but we must exclude the endpoints, because they correspond to undefined slopes.



The interval we typically associate with this arc is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

PART F: EVALUATING INVERSE TRIG FUNCTIONSThink:

A trig function such as \sin takes in angles (i.e., real numbers in its domain) as inputs and “spits out” outputs that are trig values (for \sin , values between -1 and 1 , inclusive).

On the other hand, an inverse trig function such as \sin^{-1} takes in trig values as inputs and “spits out” angles as outputs. **These angles must be in the range.**

Warning: Although calculators can provide \sin^{-1} values and other inverse trig values using degree measure, it is conventional to use radian measure, instead, since they directly correspond to “real numbers.”

Example

Evaluate $\sin^{-1}(1)$, or $\arcsin 1$.

Solution

We know that $\sin\left(\frac{\pi}{2}\right) = 1$.

Although there are other angles whose sine is 1, $\frac{\pi}{2}$ is the only one that is a “legal” output of \sin^{-1} , because it is the only one in the range of \sin^{-1} , namely $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Therefore, $\sin^{-1}(1) = \frac{\pi}{2}$.

Example

Evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, or $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$.

Solution

What angle in the range of \sin^{-1} , $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, has a sin value of $-\frac{\sqrt{2}}{2}$?

Observe that $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, so we would like a brother of $\frac{\pi}{4}$ in

Quadrant IV.

Answer: $-\frac{\pi}{4}$.

Observe that the coterminal angle $\frac{7\pi}{4}$, for example, is not in the range.

Example

Evaluate $\sin^{-1}(3)$, or $\arcsin 3$.

Solution

This is **undefined**, because 3 is not a sin value for any angle.

Observe that 3 is not in the domain of \sin^{-1} , so it is an “illegal” input.

Example

Evaluate $\cos^{-1}\left(-\frac{1}{2}\right)$, or $\arccos\left(-\frac{1}{2}\right)$.

Solution

What angle in the range of \cos^{-1} , $[0, \pi]$, has a cos value of $-\frac{1}{2}$?

Observe that $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, so we would like a brother of $\frac{\pi}{3}$ in

Quadrant II.

Answer: $\frac{2\pi}{3}$.

Example

$\tan^{-1}(10) \approx 1.47$ [radians]. Observe that $\tan^{-1}(10)$ is **not** undefined, because its domain is \mathbf{R} ; any real number is a slope (or tan value).

PART G: INVERSE PROPERTIESInverse Properties: Group 1

If x is an appropriate trig (i.e., sin, cos, tan) value, then:

$$\sin(\sin^{-1} x) = x \quad (\text{if } x \text{ is in } [-1, 1])$$

$$\cos(\cos^{-1} x) = x \quad (\text{if } x \text{ is in } [-1, 1])$$

$$\tan(\tan^{-1} x) = x \quad (\text{if } x \text{ is in } \mathbf{R})$$

Otherwise, we have “**undefined.**”

Think: “Unwrapping,” or “undoing.”

Examples

$$\cos \left(\underbrace{\cos^{-1}(0.2)}_{\text{an angle whose cosine is 0.2}} \right) = \mathbf{0.2}$$

$$\cos \left(\underbrace{\cos^{-1}(10)}_{\text{undefined}} \right) \text{ is } \mathbf{undefined}$$

$$\tan \left(\underbrace{\tan^{-1}(10)}_{\text{an angle whose tangent is 10}} \right) = \mathbf{10}$$

Inverse Properties: Group 2

If θ is in the range of the appropriate inverse trig function, then:

$$\sin^{-1}(\sin \theta) = \theta \quad \left(\text{if } \theta \text{ is in } \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \right)$$

$$\cos^{-1}(\cos \theta) = \theta \quad \left(\text{if } \theta \text{ is in } [0, \pi] \right)$$

$$\tan^{-1}(\tan \theta) = \theta \quad \left(\text{if } \theta \text{ is in } \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \right)$$

Example

$$\sin^{-1}\left(\sin \frac{\pi}{10}\right) = \frac{\pi}{10}$$

Think: The $\frac{\pi}{10}$ angle looks over at \sin^{-1} and asks, “Can you spit me out?” The \sin^{-1} function says, “Yes, I can, because you are in my range.” Also, \sin says, “ $\frac{\pi}{10}$ is in my domain, so I’m OK with that.”

Note: The “unwrapping” properties described in the box above always work for acute angles θ such as $\frac{\pi}{10}$.

Example

$$\begin{aligned}\sin^{-1}\left(\sin\frac{5\pi}{6}\right) &= \sin^{-1}\left(\frac{1}{2}\right) \\ &= \frac{\pi}{6}\end{aligned}$$

“Unwrapping” doesn’t work here, because $\frac{5\pi}{6}$ is **not** in the range of \sin^{-1} .

The especially talkative \sin^{-1} says to $\frac{5\pi}{6}$, “I can’t spit you out, but I can spit out your brother.” “When in doubt, work it out,” or ... observe that we are looking for a brother of $\frac{5\pi}{6}$ that also has a sin value of $\frac{1}{2}$ and that lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Because $\frac{1}{2} > 0$, we look in Quadrant I.

The brother we want is $\frac{\pi}{6}$.

PART H: USING RIGHT TRIANGLES TO WRITE ALGEBRAIC EXPRESSIONSExample

Write $\cot\left(\cos^{-1}\frac{x}{3}\right)$ as an equivalent algebraic expression in x .

Assume that x is such that all relevant trig and inverse trig values are defined.

Solution

Let the angle $\theta = \cos^{-1}\left(\frac{x}{3}\right)$. Think: $\cot\left(\underbrace{\cos^{-1}\frac{x}{3}}_{=\theta}\right)$

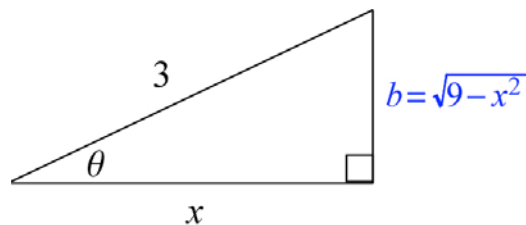
This is true $\Leftrightarrow \cos\theta = \frac{x}{3}$ and θ is in $[0, \pi]$, the range of \cos^{-1} .

We may actually assume that θ is acute, without loss of generality.

(If you are dealing with \csc^{-1} , \sec^{-1} , or \cot^{-1} , define the ranges carefully.)

Technical Note: This is a nontrivial observation! See Stewart's Precalculus book for more details.

Construct a model right triangle such that $\cos\theta = \frac{x}{3}$.



Use the Pythagorean Theorem to find an expression for the missing side length, b .

$$x^2 + b^2 = 9$$

$$b^2 = 9 - x^2$$

$$b = \pm\sqrt{9 - x^2}$$

$$b = \sqrt{9 - x^2} \quad (\text{Take the "+" root.})$$

We see that $\tan \theta = \frac{\sqrt{9-x^2}}{x}$.

Warning: $\sqrt{9-x^2} \neq 3-x$.

We then see that $\cot \theta = \frac{x}{\sqrt{9-x^2}}$, or $\frac{x\sqrt{9-x^2}}{9-x^2}$, which is our answer.

Note: Some books don't require that you rationalize the denominator.

In Calculus: This technique is used when you perform integration using trigonometric substitutions. You will see this in [Calculus II: Math 151 at Mesa](#).

SECTION 4.8: APPLICATIONS

PART A: WORD PROBLEMS

We did a word problem in [Notes 4.32](#). Some tips:

- Read the problem carefully. Underline, highlight, or summarize key pieces of information so that you don't have to keep rereading the entire problem.
- Define variables, draw and label diagrams, draw tables, etc. Visual aids often help.

In Calculus: You will need to distinguish between aspects of the problem that remain fixed (or constant) and aspects that change.

- Make appropriate unit conversions. For example, if lengths are given in both inches and feet, make sure you use only inches or only feet when you draw diagrams and write equations.
- Sometimes, the problem gives you “red herrings” – information that does not help you solve the problem. Ignore them.
- Make sure you clearly indicate the answer(s) to the question(s) you are asked, especially if your work is messy.
- Make sure you answer **all** the questions you are asked.
- Sometimes, x does not represent your final answer. You may need to use some variation, such as $x + 10$. (For example, maybe the length of a rectangle is 10 feet longer than the width (x), and you're supposed to give the length.)
- You are typically expected to give exact answers in a math class. However, on word problems, you may be asked to round off answers to a particular number of decimal places or significant digits. Avoid approximating anything until the end. You don't want to compromise the accuracy of your final answer by introducing inappropriate roundoff errors. The memory key on your calculator may be helpful. If you do round off intermediate results, use many significant digits.
- Write units where appropriate. If, for example, both feet and inches are given in a problem, make sure you know which one you have been dealing with.

- See if your answer makes sense.
- Be prepared to write your conclusions in plain English.

PART B: SOLVING RIGHT TRIANGLES

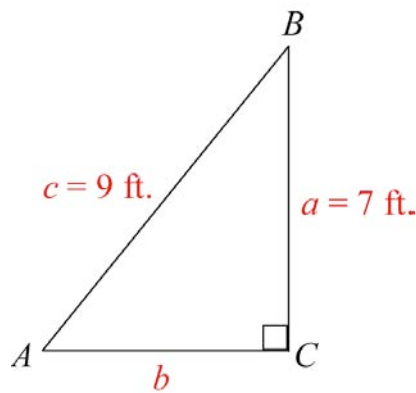
If you know two sides (i.e., side lengths) of a right triangle, or if you know one side and one of the acute angles of a right triangle, then you can “solve” the triangle, meaning that you can find all of the remaining side lengths and angles.

Tools

- The sum of the interior angles of a triangle is 180° , or π radians.
A right triangle has a right angle and two acute (and complementary) angles.
- The Pythagorean Theorem
- SOH-CAH-TOA

Example

Solve the triangle below. Round off Angles A and B to the nearest degree.



Observe that a “faces” A , b faces B , and c (the hypotenuse) faces C (the right angle). This is conventional.

Warning: If a had been given as 84 inches, convert units!

Solution

We are given a and c , and we know C is a right angle.
We must find b , A , and B .

Since we know two sides of a right triangle, we can use the Pythagorean Theorem to find the third side.

$$a^2 + b^2 = c^2$$

$$(7)^2 + b^2 = (9)^2$$

$$49 + b^2 = 81$$

$$b^2 = 32$$

$$b = \pm\sqrt{32}$$

$$\text{Take } b = \sqrt{32} \quad (\text{Take the positive root.})$$

$$b = \sqrt{16 \cdot 2}$$

$$\mathbf{b = 4\sqrt{2} \text{ ft.}} \quad (\text{Write units where appropriate.})$$

Let's find Angle A , which is an acute angle such that $\sin A = \frac{7}{9}$.

A is $\sin^{-1}\left(\frac{7}{9}\right)$. In degrees, this is about 51° .

$$\mathbf{A \approx 51^\circ}$$

Angle B is the complement of Angle A .

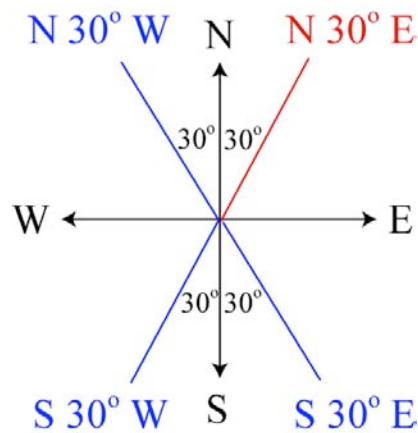
$$B \approx 90^\circ - 51^\circ$$

$$\mathbf{B \approx 39^\circ}$$

PART C: BEARING AND NAVIGATION

See [p.334](#).

An example of a bearing is $N\ 30^\circ\ E$. The first letter must be N (for due north) or S (for due south), the last letter must be W (for due west) or E (for due east), and the angle measure in the middle must be acute. $N\ 30^\circ\ E$ is read “30 degrees east of north.” The corresponding direction is a northeasterly direction (indicated in red below) that makes a 30° angle with the direction corresponding to due north. Other related directions of interest are indicated in blue.



In air navigation, bearings are measured clockwise from due north.

For example, $N\ 30^\circ\ E$ corresponds to simply 30° , but $S\ 30^\circ\ E$ corresponds to 150° :

