CHAPTER 7: SYSTEMS AND INEQUALITIES

SECTIONS 7.1-7.3: SYSTEMS OF EQUATIONS

PART A: INTRO

A solution to a system of equations must satisfy **all** of the equations in the system.

In your Algebra courses, you should have learned methods for solving systems of linear equations, such as:

$$\begin{cases} A + B = 1 \\ A - 4B = 11 \end{cases}$$

We will solve this system using both the Substitution Method and the Addition / Elimination Method in Section 7.4 on Partial Fractions.

In some cases, these methods can be extended to nonlinear systems, in which at least one of the equations is nonlinear.

PART B: THE SUBSTITUTION METHOD

See Example 1 on p.497.

Example (#8 on p.503)

Solve the nonlinear system: $\begin{cases} 3x + y = 2 \\ x^3 - 2 + y = 0 \end{cases}$

Solution

We can, for example,

(Step 1) Solve the second equation for y in terms of x and then

(Step 2) Perform a substitution into the first equation.

$$\begin{cases} 3x + y = 2 & & \Rightarrow \quad 3x + \left(2 - x^3\right) = 2 \\ x^3 - 2 + y = 0 & \Rightarrow \underbrace{y = 2 - x^3}_{\text{Call this } star.} & \nearrow \quad 3x + \cancel{2} - x^3 = \cancel{2}_0 \end{cases}$$

$$3x - x^3 = 0$$

We may prefer to rewrite this last equation so that the nonzero side has a positive leading coefficient. We're more used to that setup.

$$0 = x^3 - 3x$$

Step 3) Solve $0 = x^3 - 3x$ for x.

Warning: Remember that dividing both sides by x is risky. We may lose solutions. We prefer the Factoring method.

$$0 = x\left(x^2 - 3\right)$$

You could factor $\left(x^2 - 3\right)$ over $\mathbf{R}$ or stop factoring here.

Apply the ZFP (Zero Factor Property):

$$x = 0 \qquad \text{or} \qquad \begin{aligned} x^2 - 3 &= 0 \\ x^2 &= 3 \\ x &= \pm\sqrt{3} \end{aligned}$$

Warning: We're not done yet! We need to find the corresponding *y*-values.

Step 4) Back-substitute into *star*.

Observe that: $\left(\sqrt{3}\right)^3 = \left(\sqrt{3}\right)\left(\sqrt{3}\right)\left(\sqrt{3}\right) = 3\sqrt{3}$

x	$y = 2 - x^3$
0	$2 - (0)^3 = 2$
$\sqrt{3}$	$2 - \left(\sqrt{3}\right)^3 = 2 - 3\sqrt{3}$
$-\sqrt{3}$	$2 - \left(-\sqrt{3}\right)^3 = 2 - \left(-3\sqrt{3}\right) = 2 + 3\sqrt{3}$

Step 5) Write the solution set.

This is usually required if you are solving a system of equations.

Warning: Make sure that your solutions are written in the form (x, y), not (y, x).

The solution set here is:

$$\left\{(0, 2),\ \left(\sqrt{3}, 2 - 3\sqrt{3}\right),\ \left(-\sqrt{3}, 2 + 3\sqrt{3}\right)\right\}$$

This consists of three real solutions written as ordered pairs.
We assume that ordered pairs are appropriate, since no mention is made of *z* or other variables.

Step 6) Check your solutions in the given system. (Optional)

Warning: There is a danger in trying to check solutions in a later equivalent system that you have written down, because you may have made an error by that point. Use the original system, before you got your dirty hands all over it!

For example, we can check the solution $(0,2)$ in the given system:

$$\begin{cases} 3x + y = 2 \\ x^3 - 2 + y = 0 \end{cases}$$

Remember that a solution to a system of equations must satisfy **all** of the equations in the system.

$$\begin{cases} 3(0) + (2) = 2 \quad \Rightarrow \quad 2 = 2 \\ (0)^3 - 2 + (2) = 0 \quad \Rightarrow \quad 0 = 0 \end{cases}$$

The solution $(0,2)$ checks out.

PART C: THE GRAPHICAL METHOD

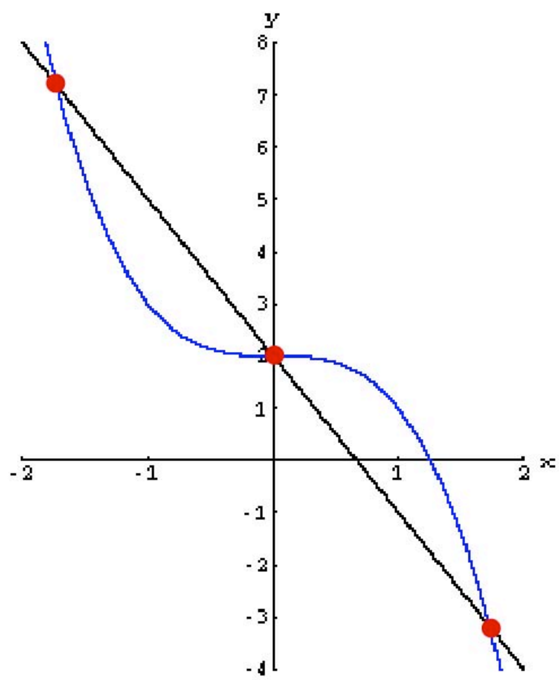
The Graphical Method for solving a system of equations requires that we graph all of the equations and then find the resulting intersection points common to **all** the graphs, if any. These points correspond to the **real** solutions to the system.

Warning: Books neglect to mention that this method is highly unreliable. For example, how can we tell visually if an intersection point is at $(0,2)$ as opposed to, say, $(0.01, 1.98)$? Although many textbook problems are designed to have "nice" solutions, we can't always assume that our solutions will only consist of integer coordinates.

Recall our Example from Part B. The solution set for the system

$$\begin{cases} 3x + y = 2 \\ x^3 - 2 + y = 0 \end{cases}$$

was $\left\{(0,2),\ \left(\sqrt{3}, 2-3\sqrt{3}\right),\ \left(-\sqrt{3}, 2+3\sqrt{3}\right)\right\}$. Consider the graphs of the two equations in the system. Based only on the figure given for #8 on p.503 (or the one below), could you have obtained the last two solutions exactly from mere visual inspection?



Note: $\left(\sqrt{3}, 2-3\sqrt{3}\right) \approx (1.7, -3.2)$, and $\left(-\sqrt{3}, 2+3\sqrt{3}\right) \approx (-1.7, 7.2)$

The figure is helpful, however, in that it seems to confirm that the system has three real solutions (corresponding to the three red intersection points), and (with the help of a calculator) the three solutions we found seem to roughly check out graphically, at least up to the limits of our vision and the precision of the figure.

The graph in black is the graph of $3x + y = 2$, which can be rewritten as $y = -3x + 2$.
The graph in blue is the graph of $x^3 - 2 + y = 0$, which can be rewritten as $y = -x^3 + 2$.

You can roughly sketch these graphs by hand, but it may be difficult to accurately locate intersection points.

Example

Find all real solutions of the system: $\begin{cases} \frac{1}{2}y = \frac{1}{2}x + 1 \\ x^2 + y^2 = 1 \end{cases}$

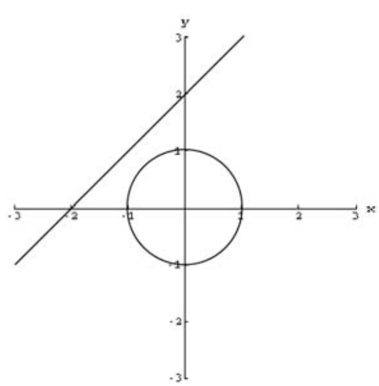
Solution

Let's multiply both sides of the first equation "through" by 2. (i.e., We multiply each term on both sides by 2.) We obtain:

$$\begin{cases} y = x + 2 \\ x^2 + y^2 = 1 \end{cases}$$

Method 1 (Graphical Method)

The first equation gives us the line below.
The second equation gives us the unit circle below.



There are no intersection points, so there are no real solutions to the system. The solution set is: $\varnothing$, the empty or null set.

Method 2 (Substitution Method)

$$\begin{cases} y = x + 2 \\ x^2 + y^2 = 1 \end{cases}$$

The first equation is already solved for y in terms of x.
Let's substitute into the second equation.

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + (x+2)^2 &= 1 \\ x^2 + x^2 + 4x + 4 &= 1 \\ 2x^2 + 4x + 4 &= 1 \\ 2x^2 + 4x + 3 &= 0 \end{aligned}$$

According to the QF (Quadratic Formula), the complex solutions of this equation are: $\dfrac{-4 \pm \sqrt{-8}}{4}$, which simplify to $\dfrac{-2 \pm i\sqrt{2}}{2}$, or $-1 \pm \dfrac{\sqrt{2}}{2} i$.

There are no viable **real** values for x, so the system has no real solutions, and the solution set is $\varnothing$.

This confirms our findings from Method 1.

Note: It is not necessary to work out the entire QF to conclude that there are no viable real values for x. It is sufficient to observe that the discriminant is negative: $b^2 - 4ac = (4)^2 - 4(2)(3) = -8 < 0$
In other problems, the discriminant can be used in conjunction with the Test for Factorability described in Section 1.5: Notes 1.48 to see if our equation can be solved quickly.

In Precalculus, a system of equations with no real solutions (i.e., an empty solution set) is called inconsistent in the sense that there is no real solution that consistently solves all of the equations in the system. The equations cannot be reconciled.

PART D: THE ADDITION / ELIMINATION METHOD

This method is based on the principle that, when you add equals to equals, you get equals. See Examples 1-3 on pp.507-509.

Example

Solve the nonlinear system: $\begin{cases} 2x + 5y^2 = 35 \\ 7x + 2y^2 = 14 \end{cases}$

Solution

The Substitution Method may get messy if we try to solve for x or y^2 in either equation.

The Addition Method may not be helpful to solve some nonlinear systems, but it is helpful here.

We can easily eliminate the y^2 terms, for example, by:

1) Multiplying "through" both sides of the first equation by 2,
2) Multiplying "through" both sides of the second equation by (-5),

and

3) Adding equals to equals to obtain a third equation that we can use to crack the system. We will informally refer to this process as adding equations.

$$\begin{cases} 2x + 5y^2 = 35 & \leftarrow \cdot (2) \\ 7x + 2y^2 = 14 & \leftarrow \cdot (-5) \end{cases}$$

In our new equivalent system, the coefficients of the y^2 terms will be opposites, and (when adding the equations) we will be able to eliminate those terms and then solve for x.

$$\begin{cases} 4x + 10y^2 = 70 \\ -35x - 10y^2 = -70 \end{cases}$$

$$\begin{aligned} -31x &= 0 \\ x &= 0 \end{aligned}$$

Warning 1: Remember to multiply the **right-hand sides** of the given equations by 2 and -5, respectively. People often focus on the left-hand sides so much that they forget about the right-hand sides.

Warning 2: Why didn't we multiply the second equation by 5 (instead of -5) and then **subtract** equals from equals? Although that would have been a correct procedure, people often make mechanical errors when subtracting. We generally prefer to **add**, instead, even if that means that we multiply both sides of an equation by a negative number. One possible exception is given in Notes 7.11. This issue will come up later in Chapter 8, when we study matrices.

Warning 3: We were lucky that the right-hand side of the resulting equation is 0, but it doesn't always have to be 0. What if you had eliminated the x terms, instead?

We can now "plug in" $x = 0$ into any of the four equations at the bottom of Notes 7.08 that contain both x and y. If we plug into the first given equation:

$$\begin{aligned} 2x + 5y^2 &= 35 \\ 2(0) + 5y^2 &= 35 \\ 5y^2 &= 35 \\ y^2 &= 7 \\ y &= \pm\sqrt{7} \end{aligned}$$

For our solutions, we require that $x = 0$, and then y can either be $\sqrt{7}$ or $-\sqrt{7}$.

Warning: Make sure you know which values of y correspond to which values of x. In Multivariable Calculus (Calculus III: Math 252 at Mesa), this issue will arise in a big way when you study Lagrange Multipliers and optimization.

Our solution set is: $\left\{\left(0, \sqrt{7}\right), \left(0, -\sqrt{7}\right)\right\}$.

Some people write $\left(0, \pm\sqrt{7}\right)$, though that may be ambiguous.

See Warning 1 in Section 1.5: Notes 1.46.

PART E: HOW MANY SOLUTIONS CAN A SYSTEM OF EQUATIONS HAVE?

We have seen nonlinear systems of equations with 0, 2, and 3 solutions. In fact, **nonlinear** systems can potentially have any whole number of solutions; they can even have infinitely many solutions. Consider the system:

$$\begin{cases} y = \sin x \\ y = 0 \end{cases}$$

However, the only possibilities for a system of **linear** equations are: 0, 1, or infinitely many solutions. Consider systems of two linear equations in two unknowns (say x and y) – and the graphs of those equations in the xy-plane. Remember that real solutions correspond to intersection points.

How many solutions?	Example	System is …
0	 different parallel lines	Inconsistent
1	 non-parallel lines	Consistent
Infinitely many	 coincident lines	Consistent

Warning: The term dependence seems to have different meanings in Introductory Algebra books and in Linear Algebra books. We will ignore this issue.

PART F: SPECIAL CASES

Example

Solve the system $\begin{cases} x + y = 2 \\ x + y = 1 \end{cases}$

Solution

Here, it is easy to "subtract" the second equation from the first.
We obtain $0 = 1$, which cannot be satisfied by any ordered pair (x, y).
In other words, there is no (x, y) for which $0 = 1$ is true.
Therefore, the system has no solutions, and the solution set is $\varnothing$.

Technical Logic Note: If this system had a solution (x, y), then $0 = 1$ would have to be true. However, we know that $0 = 1$ is not true. Therefore, the system has no solution. This is an example of indirect reasoning, which is based on the logical equivalence between an if-then statement and its contrapositive. (See Notes P.06 to P.08.)

Note: We can also say that, because $x + y = x + y$, we require that $2 = 1$ be true for any solution to the system.

We will discuss systems of linear equations with infinitely many solutions at the end of Section 8.1.

Example

How many solutions does the following system have?

$$\begin{cases} y = x + 2 \\ y = x + 2 \\ 0 = 1 \end{cases}$$

Solution

If you "subtract" the second equation from the first, then you will obtain $0 = 0$.

Warning: You may then be tempted to conclude that this system has infinitely many solutions, especially since the first two equations represent the same line in the *xy*-plane. You would be wrong!

The equation $0 = 1$ has no solutions; there is no ordered pair (x, y) such that $0 = 1$ can be satisfied. (That equation ruins it for everybody!) Therefore, the system has no solution, and the solution set is $\varnothing$.

PART G: SYSTEMS OF LINEAR EQUATIONS IN THREE UNKNOWNS (SAY *x*, *y*, and *z*)

In Section 8.1, we will solve the following system using matrices:

$$\begin{cases} 2x+2y-z= & 2 \\ x-3y+z= & -28 \\ -x+y = & 14 \end{cases}$$

In Section 7.3, a non-matrix preview of the 8.1 method is given.

We call this a system in three unknowns, even though there is no z term in the third equation.

Instead of lines in the xy-plane, we consider planes in xyz-space.

Again, the only possibilities are: 0, 1, or infinitely many solutions.
Real solutions correspond to intersection points common to **all** the planes.
See the figures on p.522.

Solutions are written as ordered triples of the form (x, y, z).
In general, when there are n unknowns, solutions are written as ordered n-tuples.

SECTION 7.4: PARTIAL FRACTIONS

PART A: INTRO

A, *B*, *C*, etc. represent unknown real constants.
Assume that our polynomials have real coefficients.

These Examples deal with rational expressions in *x*, but the methods here extend to rational expressions in *y*, *t*, etc.

Review how to add and subtract rational expressions in Section A.4: pp.A38-A39.

Review Example

$$\frac{3}{x-4}-\frac{2}{x+1}=\frac{3\overbrace{\left(x+1\right)}^{\substack{\text{Think:}\\\text{"Who's}\\\text{missing?"}}}-2\left(x-4\right)}{\left(x-4\right)\left(x+1\right)}$$

$$=\frac{x+11}{x^2-3x-4}$$

How do we reverse this process? In other words, how do we find that the partial fraction decomposition (PFD) for $\frac{x+11}{x^2-3x-4}$ is $\underbrace{\frac{3}{x-4}}_{\substack{\text{A partial}\\\text{fraction}\\\text{(PF)}}}-\frac{2}{x+1}$?

The PFD Form that we need depends on the **factored form of the denominator**. Here, the denominator is x^2-3x-4, which factors as $\left(x-4\right)\left(x+1\right)$.

PART B: THE BIG PICTURE

You may want to come back to this Part and Part C after you read the Examples starting on Notes 7.22.

In Section 2.5, we discussed:

"Factoring Over **R**" Theorem

Let $f(x)$ be a nonconstant polynomial in standard form with real coefficients. A complete factorization of $f(x)$ over **R** consists of:

1) Linear factors,
2) Quadratic factors that are **R**-irreducible (see Note below), or
3) Some product of the above, possibly including repeated factors, and
4) Maybe a nonzero constant factor.

Note: A quadratic factor is **R**-irreducible $\Leftrightarrow$
It has no real zeros (or "roots"), and it cannot be nontrivially factored and broken down further over **R** (i.e., using only real coefficients).

Knowing **how** to factor such an $f(x)$ may pose a problem, however! Finding real zeros of $f(x)$ can help you factor $f(x)$; remember the Factor Theorem from Sections 2.2 and 2.3: Notes 2.19 and 2.33.

Example

The hideous polynomial

$$6x^{14} + 33x^{13} + 45x^{12} + 117x^{11} - 213x^{10} - 2076x^{9} - 3180x^{8} - 15{,}024x^{7}$$
$$- 11{,}952x^{6} - 32{,}832x^{5} - 18{,}240x^{4} - 19{,}968x^{3} - 9216x^{2}$$

can factor over **R** as follows:

$$3x^2(x-3)(2x+1)(x+4)^2(x^2+1)(x^2+4)^3$$

Technical Note: There are other factorizations over **R** involving manipulations (like "trading") of constant factors, but we like the fact that the one provided is a factorization over **Z** (the integers), and we have no factors like $(6x+3)$ for which nontrivial GCFs (greatest common factors) can be pulled out.

Let's categorize factors in this factorization:

- 3 is a constant factor.

- We will discuss x^2 last.

- $(x-3)$ is a distinct linear factor.

  It is distinct ("different") in the sense that there are no other $(x-3)$ factors, nor are there constant multiples such as $(2x-6)$.

- $(2x+1)$ is a distinct linear factor.

- $(x+4)^2$ is a [nice] power of a linear factor. Because it can be rewritten as $(x+4)(x+4)$, it is an example of repeated linear factors.

- (x^2+1) is a distinct **R**-irreducible quadratic factor. It is irreducible over **R**, because it has no real zeros (or "roots"); it cannot be nontrivially factored and broken down further over **R**.

- $(x^2+4)^3$ is a [nice] power of an **R**-irreducible quadratic factor. Because it can be rewritten as $(x^2+4)(x^2+4)(x^2+4)$, it is an example of repeated **R**-irreducible quadratic factors.

Warning!

- x^2 actually represents repeated linear factors, because it can be rewritten as $x \cdot x$. You may want to think of it as $(x-0)^2$. We do **not** consider x^2 to be an **R**-irreducible quadratic, because it has a real zero (or root), namely 0.

Why do we care as far as PFDs are concerned?

In Notes 7.14, we showed that $\frac{x+11}{x^2-3x-4}$, or $\frac{x+11}{(x-4)(x+1)}$, can be decomposed as $\frac{3}{x-4}-\frac{2}{x+1}$. We could factor the denominator of the original expression as the product of two distinct linear factors, so we were able to decompose the expression into a sum of two rational expressions with constant numerators and linear denominators.

Note: By "sum," we really mean "sum or difference." Remember that a difference may be reinterpreted as a sum. For example, $7-4=7+(-4)$.

Every proper rational expression of the form $\frac{N(x)}{\text{nonconstant } D(x)}$,
where both $N(x)$ and $D(x)$ are polynomials in x with real coefficients,
has a PFD consisting of a sum of rational expressions ("partial fractions") whose …

… numerators can be constant or linear, and whose …

… denominators can be linear, **R**-irreducible quadratics, or powers thereof.

Note: The PFD for, say, $\frac{1}{x}$ is simply $\frac{1}{x}$. We don't really have a "sum" or a "decomposition" here.

If we have a rational expression that is improper (i.e., the degree of $N(x)$ is not less than the degree of $D(x)$), then Long Division or some other algebraic work is required to express it as either:

- a polynomial, or
- the sum of a polynomial and a proper rational expression. Think:

$$\left(polynomial\right) \; + \; \left(proper\ rational\right).$$

We then try to find a PFD for the proper rational expression.

See Notes 7.30-7.32 for an Example.

In Calculus: These PFDs are used when it is preferable (and permissible!) to apply operations (such as integration) term-by-term to a collection of "easy" fractions as opposed to a large, unwieldy fraction. The PFD Method for integration (which is the reverse of differentiation, the process of finding a derivative) is a key topic of Chapter 9 in the Calculus II: Math 151 textbook at Mesa. As it turns out, the comments above imply that we can integrate **any** rational expression up to our ability to factor polynomial denominators. This is a **very** powerful statement!

PART C: PFD FORMS

Let $r(x)$ be a proper rational expression of the form $\dfrac{N(x)}{\text{nonconstant } D(x)}$,

where both $N(x)$ and $D(x)$ are polynomials in x with real coefficients.

Consider a complete factorization of $D(x)$ over **R**. (See Part B.)

Each **linear** or **R-irreducible quadratic** factor of $D(x)$ contributes a term (a partial fraction) to the PFD Form.

Let $m, a, b, c \in \mathbf{R}$.

Category 1a: Distinct Linear Factors; form $(mx+b)$

$(mx+b)$ contributes a term of the form:

$$\frac{A}{mx+b} \quad (A \in \mathbf{R})$$

Note: We may use letters other than A.

Category 1b: Repeated (or Powers of) Linear Factors; form $(mx+b)^n$, $n \in \mathbf{Z}^+$

$(mx+b)^n$ contributes a sum of n terms:

$$\frac{A_1}{mx+b} + \frac{A_2}{(mx+b)^2} + \ldots + \frac{A_n}{(mx+b)^n} \quad (\text{each } A_i \in \mathbf{R})$$

Warning / Think: "Run up to the power." Also observe that each term gets a **numerator** of **constant form**.

Technical Note: $(x+2)$ and $(3x+6)$ do **not** count as distinct linear factors, because they are only separated by a constant factor (3), which can be factored out of the latter.

Note: You can think of Category 1a as a special case of this where $n=1$.

Category 2a: Distinct **R**-Irreducible Quadratic Factors; form $\left(ax^2 + bx + c\right)$

$\left(ax^2 + bx + c\right)$ contributes a term of the form:

$$\frac{Ax + B}{ax^2 + bx + c} \quad \left(A, B \in \mathbf{R}\right)$$

Category 2b: Repeated (or Powers of) **R**-Irreducible Quadratic Factors; form $\left(ax^2 + bx + c\right)^n$, $n \in \mathbf{Z}^+$

$\left(ax^2 + bx + c\right)^n$ contributes a sum of n terms:

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{\left(ax^2 + bx + c\right)^2} + \ldots + \frac{A_n x + B_n}{\left(ax^2 + bx + c\right)^n} \quad \left(\text{each } A_i, B_i \in \mathbf{R}\right)$$

Warning / Think: "Run up to the power." Also observe that each term gets a **numerator** of **linear form**, though the numerator may turn out to be just a constant.

Note: You can think of Category 2a as a special case of this where $n = 1$.

Example

Find the PFD Form for $\frac{1}{x^2(x-4)^2(x^2+1)}$.

You do not have to solve for the unknowns.

Solution

$$\frac{1}{x^2(x-4)^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4} + \frac{D}{(x-4)^2} + \frac{Ex+F}{x^2+1}$$

$$(A, B, C, D, E, F \in \mathbf{R})$$

Warning: Remember that x^2 represents repeated linear factors.
See Notes 7.16.

We "run up to the power" for both the x^2 and $(x-4)^2$ factors in the denominator.

Because x^2+1 is an **R**-irreducible quadratic, we have a linear form, $Ex+F$, in the corresponding numerator.

Technical Note: If a constant aside from ± 1 can be factored out of the denominator, you can do so immediately. A factor of 3 in the denominator, for example, can be factored out of the overall fraction as a $\frac{1}{3}$. This may help, because you do not want to consider $(x+2)$ and $(3x+6)$, for example, as distinct linear factors. We prefer **complete** factorizations over **R**.

PART D: STEPS; DISTINCT LINEAR FACTORS

Example

Find the PFD for $\dfrac{x+11}{x^2-3x-4}$. (Let's reverse the work from Part A.)

Solution

Step 1: If the expression is improper, use Long Division to obtain the form: (polynomial) + (proper rational expression).

Technical Note: Synthetic Division works when the denominator is of the form $x-k,\ k\in\mathbf{R}$. In that case, you wouldn't need a PFD!

$\dfrac{x+11}{x^2-3x-4}$ is proper, so Long Division is unnecessary.

Step 2: Factor the denominator (completely) over **R**.

$$\frac{x+11}{x^2-3x-4}=\frac{x+11}{(x-4)(x+1)}$$

Step 3: Determine the required PFD Form.

The denominator consists of distinct linear factors, so the PFD form is given by:

$$\frac{x+11}{(x-4)(x+1)}=\frac{A}{x-4}+\frac{B}{x+1}$$

Note: The form $\dfrac{A}{x+1}+\dfrac{B}{x-4}$ may also be used. The roles of A and B will then be switched in the following work.

Step 4: Multiply both sides of the equation by the LCD (least or lowest common denominator), the denominator on the left.

These steps can be skipped:

$$\cancel{(x-4)(x+1)}\left[\frac{x+11}{\cancel{(x-4)(x+1)}}\right]=\cancel{(x-4)}(x+1)\left[\frac{A}{\cancel{x-4}}\right]+(x-4)\cancel{(x+1)}\left[\frac{B}{\cancel{x+1}}\right]$$

Instead, we can use the "Who's missing?" trick for each term on the right side:

$$x+11=A(x+1)+B(x-4)$$

This is called the basic equation.

Step 5: Solve the basic equation for the unknowns, *A* and *B*.

Technical Note: *A* and *B* are unique, given the PFD Form. The PFD will be unique up to a reordering of the terms and manipulations of constant factors.

Method 1 ("Plug In"): Plug convenient values for *x* into the basic equation.

For the correct values of *A* and *B*, the basic equation holds true for **all** real values of *x*, even those values excluded from the domain of the original expression. This can be proven in Calculus.

We would like to choose values for *x* that will make the "coefficient" of *A* or *B* equal to 0.

Plug in $x=-1$:

$$x+11=A(x+1)+B(x-4)$$
$$-1+11=\cancelto{0}{A(-1+1)}+B(-1-4)$$
$$10=-5B$$
$$\boldsymbol{B=-2}$$

Plug in $x = 4$:

$$x + 11 = A(x+1) + B(x-4)$$
$$4 + 11 = A(4+1) + \cancel{B(4-4)}^{0}$$
$$15 = 5A$$
$$\boldsymbol{A = 3}$$

Note: Other values for *x* may be chosen, but you run the risk of having to solve a more complicated system of linear equations.

Method 2 ("Match Coefficients"): Write the right-hand side of the basic equation in standard form, and match (i.e., equate) corresponding coefficients.

$$x + 11 = A(x+1) + B(x-4)$$
$$x + 11 = Ax + A + Bx - 4B$$
$$(1)x + (11) = (A+B)x + (A-4B)$$

The "(1)" coefficient and the parentheses on the left side are optional, but they may help you clearly identify coefficients.

Given that both sides are written in standard form,
the left-hand side is equivalent to the right-hand side $\Leftrightarrow$
every corresponding pair of coefficients of like terms are equal.

We must solve the system:

$$\begin{cases} A + B = 1 \\ A - 4B = 11 \end{cases}$$

See Sections 7.1 and 7.2 for a review of how to solve systems of two linear equations in two unknowns.

We could solve the first equation for A and use the Substitution Method:

$$\begin{cases} A + B = 1 & \Rightarrow \quad A = 1 - B \quad \searrow \\ A - 4B = 11 & \qquad\qquad\qquad \Rightarrow \quad (1 - B) - 4B = 11 \end{cases}$$

$$1 - 5B = 11$$

$$-10 = 5B$$

$$\boldsymbol{B = -2}$$

Then,

$$A = 1 - B$$

$$A = 1 - (-2)$$

$$\boldsymbol{A = 3}$$

Alternately, we could multiply both sides of the first equation by -1 (so that we have opposite coefficients for one of the unknowns) and use the Addition / Elimination Method, in which we "add equations" (really, add equals to equals) to obtain a new equation:

$$\begin{cases} A + B = 1 & \leftarrow \cdot (-1) \\ A - 4B = 11 \end{cases}$$

$$\begin{cases} -A - B = -1 \\ A - 4B = 11 \end{cases}$$

$$-5B = 10$$

$$\boldsymbol{B = -2}$$

Now, let's use the original first equation to find A:

$$A + B = 1$$

$$A + (-2) = 1$$

$$\boldsymbol{A = 3}$$

Note: We may want to combine Methods 1 and 2 ("Plug In" and "Match Coefficients") when solving more complicated problems. Method 1 is usually easier to use, so we often use it first to find as many of the unknowns as we can with ease. (Remember that **any** real value for x may be plugged into the basic equation.) We can then plug values of unknowns we have found into the basic equation and use Method 2 to find the values of the remaining unknowns. Method 2 tends to be more directly useful as we discuss more complicated cases.

Step 6: Write out the PFD.

$$\frac{x+11}{x^2-3x-4} \quad \text{or} \quad \frac{x+11}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1}$$

$$= \frac{3}{x-4} + \frac{-2}{x+1}$$

$$= \frac{3}{x-4} - \frac{2}{x+1}$$

PART E: REPEATED (OR POWERS OF) LINEAR FACTORS

Example

Find the PFD for $\frac{x^2}{(x+2)^3}$.

Solution

Step 1: The expression is proper, because 2, the degree of $N(x)$, is less than 3, the degree of $D(x)$.

Warning: Imagine that $N(x)$ and $D(x)$ have been written out in standard form before determining the degrees.

Step 2: Factor the denominator over **R**. (Done!)

Step 3: Determine the required PFD Form.

The denominator consists of repeated linear factors, so the PFD Form is given by:

$$\frac{x^2}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

We must "run up to the power."

Step 4: Multiply both sides of the equation by the LCD, $(x+2)^3$, to obtain the basic equation.

$$x^2 = A(x+2)^2 + B(x+2) + C$$

<u>Step 5</u>: Solve the basic equation for the unknowns, *A*, *B*, and *C*.

Plug in $x = -2$:

$$x^2 = A(x+2)^2 + B(x+2) + C$$

$$(-2)^2 = \cancel{A(-2+2)^2}^{\,0} + \cancel{B(-2+2)}^{\,0} + C$$

$$\mathbf{C = 4}$$

Updated basic equation; we now know $C = 4$:

$$x^2 = A(x+2)^2 + B(x+2) + 4$$

We can use the "Match Coefficients" Method, or we can plug in a couple of other real values for *x*, as follows:

Plug in $x = 0$, say:

$$x^2 = A(x+2)^2 + B(x+2) + 4$$

$$(0)^2 = A(0+2)^2 + B(0+2) + 4$$

$$0 = 4A + 2B + 4$$

Let's divide both sides by 2 and switch sides.

$$2A + B + 2 = 0$$

$$2A + B = -2$$

Plug in $x = -1$, say:

$$x^2 = A(x+2)^2 + B(x+2) + 4$$

$$(-1)^2 = A(-1+2)^2 + B(-1+2) + 4$$

$$1 = A + B + 4$$

$$-3 = A + B$$

$$A + B = -3$$

We must solve the system:

$$\begin{cases} 2A + B = -2 \\ A + B = -3 \end{cases}$$

After some work, we find that $\boldsymbol{A = 1}$ and $\boldsymbol{B = -4}$.

Step 6: Write out the PFD.

$$\begin{aligned} \frac{x^2}{(x+2)^3} &= \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} \\ &= \frac{1}{x+2} + \frac{-4}{(x+2)^2} + \frac{4}{(x+2)^3} \\ &= \frac{1}{x+2} - \frac{4}{(x+2)^2} + \frac{4}{(x+2)^3} \end{aligned}$$

PART F: DISTINCT R-IRREDUCIBLE QUADRATIC FACTORS

Example

$\frac{1}{x^2+1}$ can't be decomposed further using a PFD over **R**, because the denominator is, itself, an irreducible quadratic over **R** (i.e., it has no real roots).

Technical Note: There is such a thing as a PFD over **C**.

Example

Find the PFD for $\frac{2x^4-2x^3+10x^2-3x+9}{2x^3+3x}$.

Solution

Step 1: The expression is improper, because 4, the degree of $N(x)$, is not less than 3, the degree of $D(x)$.

After performing the Long Division (left up to you!), we obtain:

$$\frac{2x^4-2x^3+10x^2-3x+9}{2x^3+3x} = x-1+\frac{7x^2+9}{2x^3+3x}$$

Step 2: Factor the denominator over **R**.

Warning: We will ignore the polynomial part, $x-1$, for now, but don't forget about it when giving your final answer.

$$\frac{7x^2+9}{2x^3+3x} = \frac{7x^2+9}{x(2x^2+3)}$$

Step 3: Determine the required PFD Form.

$$\frac{7x^2+9}{x\left(2x^2+3\right)} = \frac{A}{x} + \frac{Bx+C}{2x^2+3}\text{; remember } x-1$$

You should remind yourself of the $x-1$ polynomial part here, because you will refer to this step when you write out the PFD at the end.

Step 4: Multiply both sides of the equation by the LCD, $x\left(2x^2+3\right)$, to obtain the basic equation.

$$7x^2+9 = A\left(2x^2+3\right) + \left(Bx+C\right)x$$

Step 5: Solve the basic equation for the unknowns, A, B, and C.

Plug in $x=0$:

$$7x^2+9 = A\left(2x^2+3\right) + \left(Bx+C\right)x$$

$$7(0)^2+9 = A\left[2(0)^2+3\right] + \cancel{\left[B(0)+C\right](0)}^{\,0}$$

$$9 = 3A$$

$$\boldsymbol{A=3}$$

Updated basic equation; we now know $A=3$:

$$7x^2+9 = 3\left(2x^2+3\right) + \left(Bx+C\right)x$$

Let's use the "Match Coefficients" Method.

$$7x^2 \color{red}{+0x} + 9 = 6x^2+9+Bx^2+Cx$$

$$(7)x^2+(0)x+(9) = (6+B)x^2+(C)x+(9)$$

Tip: Inserting the $+0x$ on the left side may be helpful.

Note: The "9"s better match up, or else we're in trouble!

Match the x^2 coefficients:

$$7 = 6 + B$$
$$\boldsymbol{B = 1}$$

Match the x coefficients:

$$0 = C$$
$$\boldsymbol{C = 0}$$

Step 6: Write out the PFD.

Warning: Don't forget the polynomial part, $x-1$. That's why we had that reminder back in Step 3.

$$\begin{aligned}\frac{2x^4 - 2x^3 + 10x^2 - 3x + 9}{2x^3 + 3x} &= x - 1 + \frac{A}{x} + \frac{Bx + C}{2x^2 + 3} \\ &= x - 1 + \frac{3}{x} + \frac{1x + 0}{2x^2 + 3} \\ &= x - 1 + \frac{3}{x} + \frac{x}{2x^2 + 3}\end{aligned}$$

PART G: REPEATED (OR POWERS OF) R-IRREDUCIBLE QUADRATIC FACTORS

Example

Find the PFD for $\dfrac{2x^3 - x^2 + 2x + 2}{\left(x^2 + 1\right)^2}$.

Solution

Step 1: The expression is proper, because 3, the degree of $N(x)$, is less than 4, the degree of $D(x)$.

Warning: Imagine that $N(x)$ and $D(x)$ have been written out in standard form before determining their degrees.

Step 2: Factor the denominator over **R**. (Done!)

Observe that $\left(x^2 + 1\right)$ has no real zeros.

Step 3: Determine the required PFD Form.

$$\frac{2x^3 - x^2 + 2x + 2}{\left(x^2 + 1\right)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{\left(x^2 + 1\right)^2}$$

We must "run up to the power."

Step 4: Multiply both sides of the equation by the LCD, $\left(x^2 + 1\right)^2$, to obtain the basic equation.

$$2x^3 - x^2 + 2x + 2 = (Ax + B)\left(x^2 + 1\right) + (Cx + D)$$

Step 5: Solve the basic equation for the unknowns, *A*, *B*, *C*, and *D*.

Let's use the "Match Coefficients" Method immediately.

$$2x^3 - 1x^2 + 2x + 2 \;=\; Ax^3 + Ax + Bx^2 + B + Cx + D$$
$$(2)x^3 + (-1)x^2 + (2)x + (2) = (A)x^3 + (B)x^2 + (A + C)x + (B + D)$$

Note: If you use the optional parentheses on the left side, remember to separate your terms with "+" signs in order to avoid confusion.

Note: You may want to mark off terms as you collect like terms on the right side.

By matching the x^3 coefficients and the x^2 coefficients, we immediately obtain:

$$\boldsymbol{A = 2}$$
$$\boldsymbol{B = -1}$$

Match the x coefficients, and use the previous info:

$$\begin{aligned} A + C &= 2 \\ 2 + C &= 2 \\ \boldsymbol{C} &\boldsymbol{= 0} \end{aligned}$$

Match the constant terms, and use the previous info:

$$\begin{aligned} B + D &= 2 \\ -1 + D &= 2 \\ \boldsymbol{D} &\boldsymbol{= 3} \end{aligned}$$

We have solved the system:

$$\begin{cases} A = 2 \\ B = -1 \\ A + C = 2 \\ B + D = 2 \end{cases}$$

Step 6: Write out the PFD.

$$\frac{2x^3 - x^2 + 2x + 2}{\left(x^2+1\right)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{\left(x^2+1\right)^2}$$

$$= \frac{2x-1}{x^2+1} + \frac{0x+3}{\left(x^2+1\right)^2}$$

$$= \frac{2x-1}{x^2+1} + \frac{3}{\left(x^2+1\right)^2}$$

Historical Note on p.536: John Bernoulli introduced the methods of this section.

SECTION 7.5: GRAPHING INEQUALITIES, and SECTION 7.6: LINEAR PROGRAMMING

In Example 2 on p.542, the one-variable linear inequalities $x > -2$ and $y \leq 3$ are graphed in the *xy*-plane.

In Example 3 on p.542, the two-variable linear inequality $x - y < 2$ is graphed in the *xy*-plane. Two methods:

Method 1: Test Point Method

Step 1: Graph the boundary line, which separates the *xy*-plane into two half-planes.

• Replace the inequality symbol with "=" to obtain the equation of the boundary line.

• To figure out how to graph the line,

§ Put the equation in slope-intercept form: $y = mx + b$, or

§ Plot the intercepts. (See Section 1.3: Notes 1.16-1.17.)

•

If the inequality had	Then graph the line as
$\leq$ or $\geq$ (weak inequality)	a solid line (We include the line in the graph.)
$<$ or $>$ (strict inequality)	a dashed line (We exclude the line.)

Step 2: Decide which half-plane to shade.

• Pick a test point not on the boundary line. $(0,0)$ is usually the best choice if it doesn't lie on the line.

• If the coordinates of the test point make the inequality true, shade the half-plane containing the test point (i.e., shade "towards" the test point). Otherwise, shade the other half-plane (i.e., shade "away from" the test point).

Method 2: "Solve for *y*" Method

Step 1: Put the inequality in the form

$$y \begin{pmatrix} \text{inequality} \\ \text{symbol} \\ <,\ \leq,\ >,\ \text{or}\ \geq \end{pmatrix} mx+b$$

Step 2: Graph the boundary line, whose equation is $y = mx + b$.
(The box from Method 1 on solid vs. dashed applies here, too.)

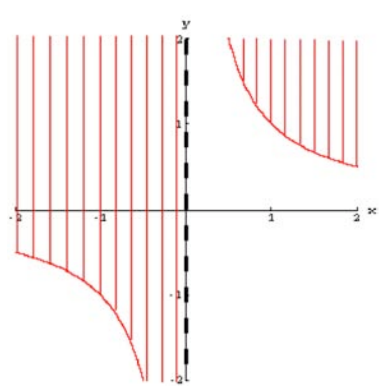
Step 3: If the inequality from Step 1 is of the form:

$$\left.\begin{matrix} y > mx+b \\ y \geq mx+b \end{matrix}\right\} \text{favor "high" points, so shade } \textbf{above} \text{ the boundary line}$$

$$\left.\begin{matrix} y < mx+b \\ y \leq mx+b \end{matrix}\right\} \text{favor "low" points, so shade } \textbf{below} \text{ the boundary line}$$

In Example 1 on p.541, the nonlinear inequality $y \geq x^2 - 1$ is graphed in the *xy*-plane. The graph of $y = x^2 - 1$ divides the *xy*-plane into two regions. To be safe, because the inequality is nonlinear, we use a test point in **each** of the two regions to see if none, one, or both of the regions should be shaded. Notice that the inequality favors "high" points.

Warning: If we graph the inequality $y \geq \frac{1}{x}$, the graphs of $y = \frac{1}{x}$ **and** $x = 0$ are used to separate the *xy*-plane into regions. Observe that $\frac{1}{x}$ is undefined when $x = 0$.



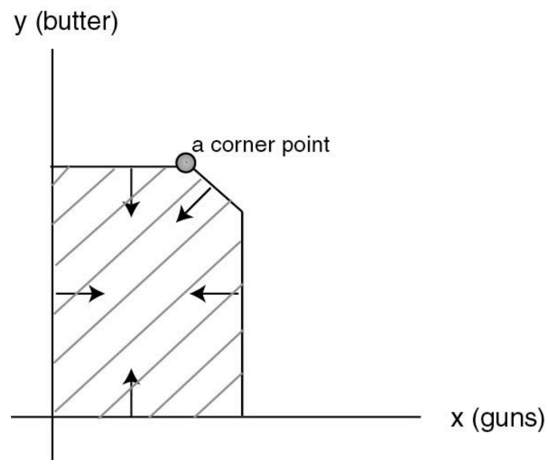
Systems of inequalities are discussed on pp.543-547. As with systems of equations, we want to locate all points that represent solutions to **all** the inequalities in the system. In other words, we want to locate the points where **all** the graphs of the inequalities intersect (or "overlap").

Linear programming is the key application of systems of linear inequalities.

Let's say we want to produce guns and/or butter. (In a way, this was the issue facing President Lyndon Johnson when he was fighting both the Vietnam War and the War on Poverty.) Given limited budget and/or physical constraints, what are our production options? That is, what combinations of guns and butter can we produce? The possible combinations make up the feasible region shaded below; this feasible region represents the solution set of a system of linear inequalities (corresponding to the constraints).

The inequalities must be weak inequalities; i.e., they must contain $\leq$ or $\geq$ symbols, as opposed to $<$ or $>$ symbols.

Notice that we have the nonnegativity constraints $x \geq 0$ and $y \geq 0$.



What is the "best" production strategy? We first need an objective function that assigns a value to each of the points in the feasible region; these values are a measure of "goodness" or "badness," such as profit or cost. We want the point in the feasible region that has the maximum or minimum such value. This point represents our optimal solution – that is, our best production strategy. If everything's linear (the objective function and the constraints), then the optimal solution will lie at a corner point (or take up some boundary) of the feasible region. These ideas extend to higher dimensions! See Notes 7.40.

<u>Example</u>

Objective function $z = f(x, y)$

$z = 4x + 3y$, where:

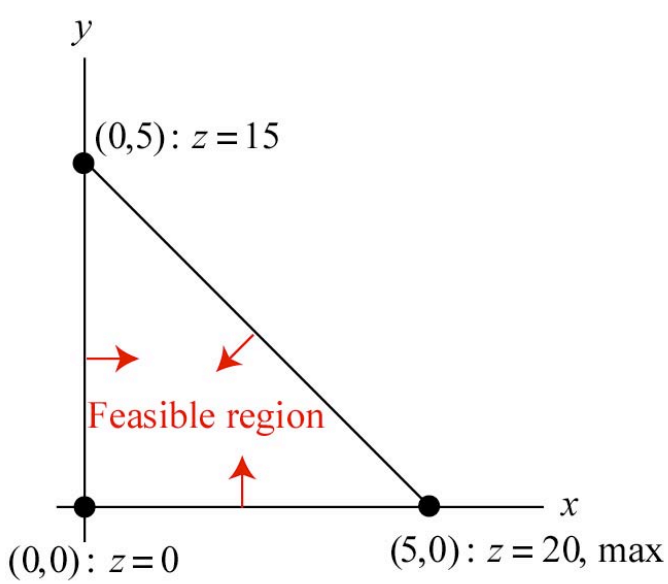
z represents profit that we want to maximize
x and y represent quantities of two products

Constraints

$x \geq 0$
$y \geq 0$
$x + y \leq 5$ (Total production cannot exceed 5 pounds, for example.)

Feasible region

The optimal solution must occur at a corner point.
That corner point turns out to be $(5,0)$.



How has linear programming been used?

- In scheduling, transportation, and resource allocation applications during World War II

- In the famous Berlin Airlift (1948-1949) during the Soviet blockade of Berlin

- In the 1980s, American Airlines cut costs 10% using linear programming techniques.

- In pricing and supply management

- It's the basis for 10% to 25% of all scientific computation today.

Oh, and that urban legend about that brilliant graduate student who solved two problems he saw on the blackboard, thinking that they were homework problems (they were actually as-yet unsolved research problems) – that's true. That student was George Dantzig, who invented the simplex method in 1947. It is the key algorithm used in linear programming. The simplex method allows us to efficiently travel along the edges of a multidimensional polytope bounding the feasible region.

See Larson, p.554 to see a photo of Dantzig, who died in May 2005.

Sources:

Larson, "linear programming" from *Encyclopedia Britannica*,
Stanford Report (May 25, 2005) link:
http://news-service.stanford.edu/news/2005/may25/dantzigobit-052505.html