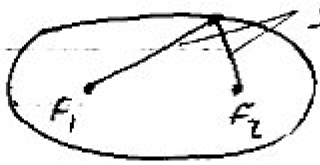
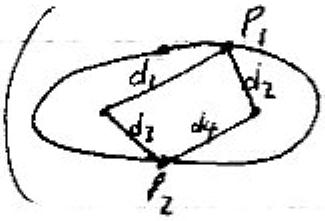
10.3: ELLIPSES

Ⓐ Locus Definition (of an Ellipse)

fo-sigh or fo-kigh

Idea Two fixed points (foci).

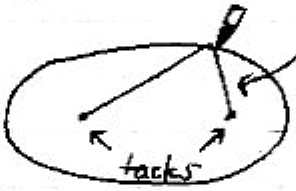




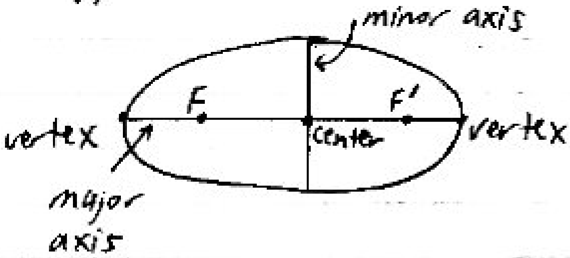
$$\left(d_1 + d_2 = d_3 + d_4 \right)$$

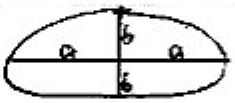
Constructing an Ellipse

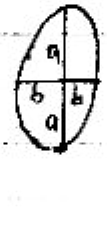
rubber band (inelastic) string stays same length



Ⓑ Terminology







We'll assume $0 < b < a$

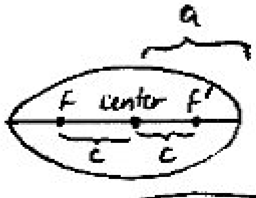
(i.e., $a > b$ — both "+")

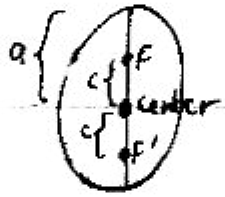
major axis $= 2a$

minor axis $= 2b$

$a = b = r$

If $a = b \to$ circle





$c^2 = a^2 - b^2$

Notice: $c < a$

(We'll assume $0 < b < a$.

i.e., $a > b$

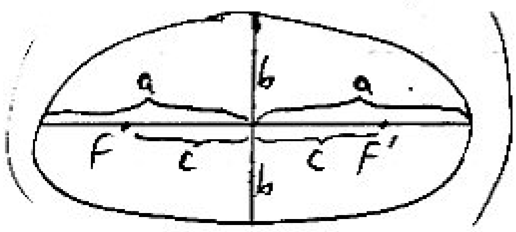
both "+"

If $a = b \rightarrow$ circle $\oplus$)

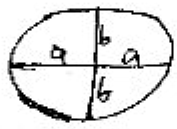
© If the Center is (0,0)

① $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

larger denom. under x^2

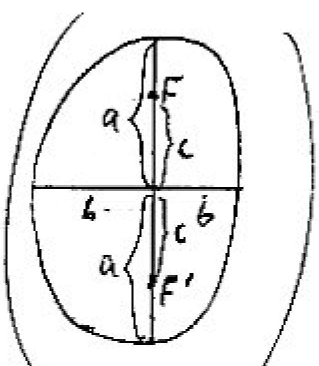


"x-long"

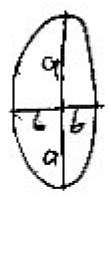


② $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$

larger denom. under y^2



"y-long"



D) If the Center is (h,k)

① → $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

② → $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

To quote Dennis Hopper

E) Eccentricity "e"

$e = \frac{c}{a}$

Circle

$e=0$ $\quad$ $e=0.3$ $\quad$ $e=0.8$

Ellipse: $\left(\begin{array}{l} 0 < c < a \\ 0 < \frac{c}{a} < 1 \end{array}\right)$

$0 < e < 1$

F) Ex (#16)

Graph $9x^2 + 4y^2 - 54x + 40y + 37 = 0$
Find the center, vertices, foci, and e.

Group terms.

$$\underbrace{(9x^2 - 54x)}_{\text{Factor out leading coeff.} = 9} + \underbrace{(4y^2 + 40y)}_{\text{Factor out } 4} = -37$$

$$9(x^2 - 6x + 9) + 4(y^2 + 10y + 25) = -37 + \underbrace{9(9) + 4(25)}_{\text{to be fair}}$$

CTS   CTS

Factor!

$$9(x-3)^2 + 4(y+5)^2 = 144$$

Need a "1"
$\div 144$

What if we had 75 instead of 144?

$$\frac{9(x-3)^2}{144} + \frac{4(y+5)^2}{144} = 1$$

$$\frac{(x-3)^2}{\frac{144}{9}} + \frac{(y+5)^2}{\frac{144}{4}} = 1$$

$$\frac{(x-3)^2}{16} + \frac{(y+5)^2}{36} = 1$$

Center: $(3, -5)$
makes left side $= 0$

("Like" $\frac{x^2}{16} + \frac{y^2}{36} = 1$) (0 y-long)

$b^2 = 16$ $\quad a^2 = 36$ (larger denom.)

$b = 4$ $\quad a = 6$ (take "+" root)

<u>Find c</u>

$$c^2 = a^2 - b^2$$
$$= 36 - 16$$
$$= 20$$
$$c = \sqrt{20}$$
$$= 2\sqrt{5}$$

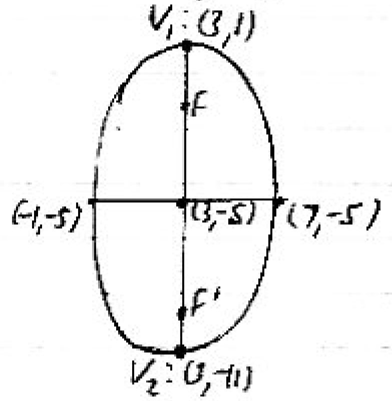
<u>Find e</u>

$$e = \frac{c}{a}$$
$$= \frac{2\sqrt{5}}{6}$$
$$= \frac{\sqrt{5}}{3} \approx 0.745$$

<u>Sketch the graph</u>

$a = 6$
$b = 4$
$c = 2\sqrt{5}$

"y stuff" has larger denom. (a^2), so y-long 0



$F: (3, -5+2\sqrt{5})$
$F': (3, -5-2\sqrt{5})$

can wait...

⑥ Applications

Reflecting Property

Pool Table

Statuary Hall in U.S. Capitol

J.Q. Adams eavesdropped.

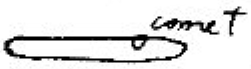
what happens surrounding tissue

Lithotripsy - destroying kidney stones

Kepler's Laws of Planetary Motion (1609)

Small effect on temp. True!

Copernicus asked to revise calendar - not accurate w/ geocentric



(can have parabolic, hyperbolic orbit)

equal areas in equal times

Larson 646

Comets disc. before 1970
48% parabolic
40% ell.
11% hyp.

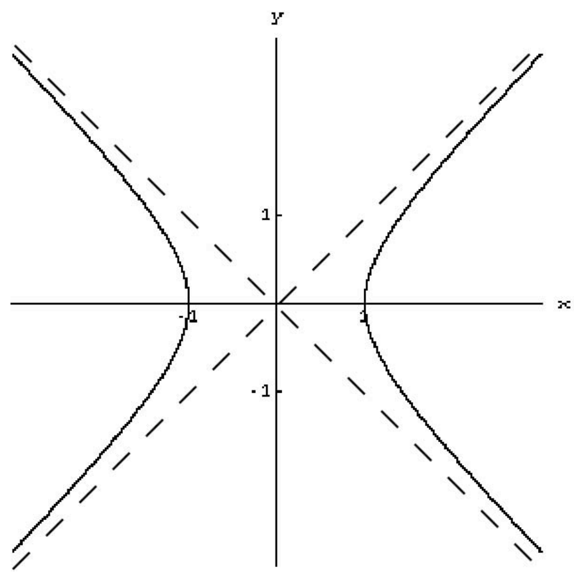
Halley $e = 0.97$
Kohoutek $\approx .999925$

SECTION 10.4: HYPERBOLAS

Hyperbolas have two branches. (They may be considered as part of a single branch, if you allow the branches to pass through the point at infinity. Think of a baseball. Don't worry about this for now!)

Technical Note: The locus definition of hyperbolas is similar to the one for ellipses, except that, instead of the sum of the distances to the two foci being kept constant, it is the absolute value of the difference of the distances.

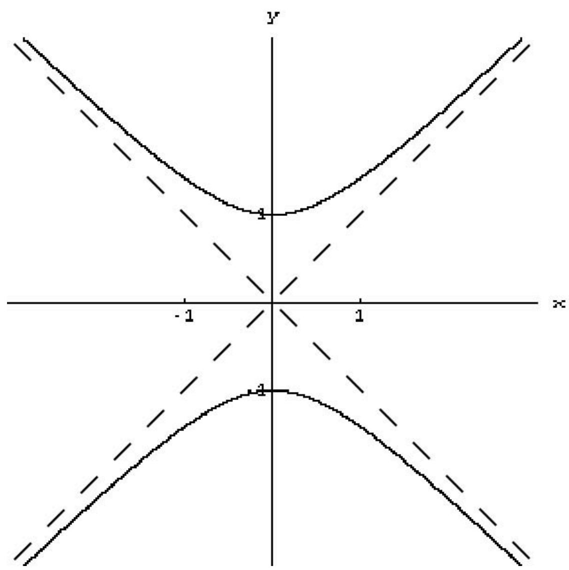
The graph of $x^2 - y^2 = 1$ is below:



The equation confirms that ± 1 are *x*-intercepts of the graph (plug in $y = 0$), but there are no *y*-intercepts (plug in $x = 0$ to see why).

The dashed lines are the asymptotes of the hyperbola. The two branches approach those lines but never cross them.

The graph of $y^2 - x^2 = 1$ is below:



The equation confirms that ± 1 are *y*-intercepts of the graph (plug in $x = 0$), but there are no *x*-intercepts (plug in $y = 0$ to see why).

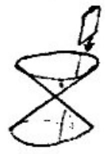
It helps to remember that:

- The graph of $x^2 - y^2 = 1$ "opens" along the *x*-axis, while

- The graph of $y^2 - x^2 = 1$ "opens" along the *y*-axis.

10,4: HYPERBOLAS

have 2 branches)(



Manip.

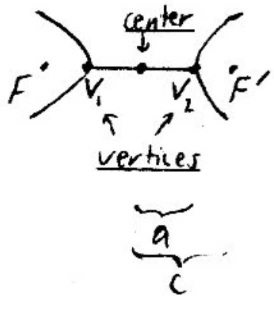
(A) Locus Def'n

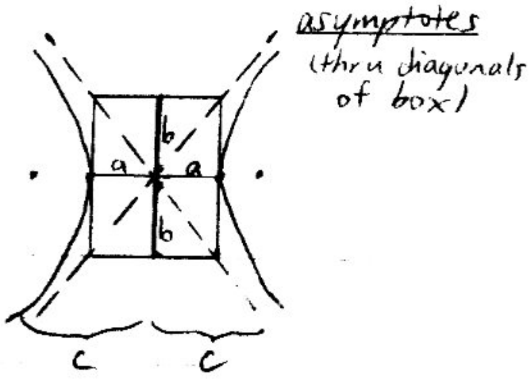
In an ellipse...

2 foci

(B) Terminology

Some of you are





Transverse axis $= 2a$

Conjugate axis $= 2b$

$c^2 = a^2 + b^2$

Notice: $c > a$

Scarface
1st you draw the box...

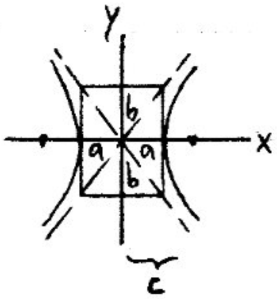
Draw: ① Box
② Asymptotes
③ Hyperbola

Ⓒ If ___ Center is (0,0)

① $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

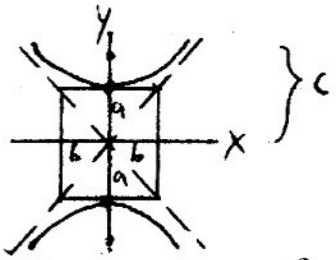
a^2 on left
(not necessarily $> b^2$)

$\frac{\text{rise}}{\text{run}}$



Asyms: $y = \underbrace{\pm \frac{b}{a}}_{\text{slopes}} x$

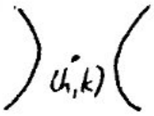
② $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$



Asyms: $y = \pm \frac{a}{b} x$

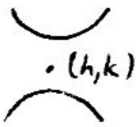
Ⓓ If ... Center is (h,k)

① ⇒ $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$



Asyms:
$y - k = \pm \frac{b}{a}(x-h)$

② ⇒ $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$



Asyms:
$y - k = \pm \frac{a}{b}(x-h)$

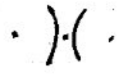
Ⓔ Eccentricity

$e = \frac{c}{a}$

Is $c>a$ or $c<a$?

$c > a$, so $e > 1$

$e = 2$



$e = 10$

(F) Ex (#14)

Graph $y^2 - 4x^2 - 12y - 16x + 16 = 0$
Find center, vertices, foci, and asymptotes.

Group terms

$$(y^2 - 12y) + \underbrace{(-4x^2 - 16x)}_{\text{Factor out } (-4)} = -16$$

$$(y^2 - 12y \underbrace{+36}_{CTS}) - 4(x^2 + 4x \underbrace{+4}_{CTS}) = -16 \underbrace{+36 - 4(4)}_{\text{Be fair!}}$$

Factor!

$$(y-6)^2 - 4(x+2)^2 = 4$$

Need a "1"
÷ thru by 4

$$\frac{(y-6)^2}{4} - \frac{(x+2)^2}{1} = 1$$

$$\boxed{\text{Center: } (\overset{x}{\underset{h}{-2}}, \overset{y}{\underset{k}{6}})}$$

"y stuff" on left, so $a^2 = 4$

$a^2 = 4$ $\quad$ $b^2 = 1$

$a = \pm 2$ $\quad$ $b = \pm 1$

Find c

$$c^2 = a^2 + b^2$$
$$= 4 + 1$$
$$= 5$$

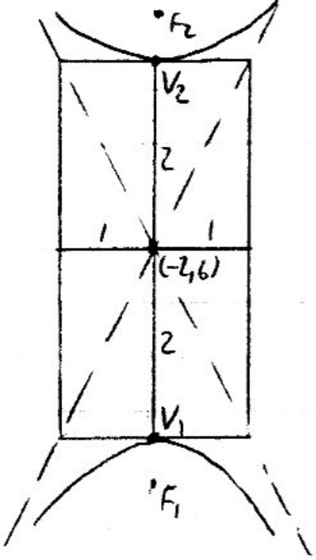
$$c = \sqrt{5}$$

Asyms

$$y - k = \pm \frac{a}{b}(x - h)$$
$$\left(y - 6 = \pm \frac{2}{1}(x - (-2))\right)$$
$$\boxed{y - 6 = \pm 2(x + 2)}$$

Graph Setup

$a = 2,\ b = 1,\ c = \sqrt{5} \approx 2.2$



V_1: $(-2, 6-2)$
$\boxed{(-2, 4)}$

V_2: $(-2, 6+2)$
$\boxed{(-2, 8)}$

F_1: $\boxed{(-2, 6-\sqrt{5})}$
F_2: $\boxed{(-2, 6+\sqrt{5})}$

LORAN
Intersect hyps ⇒ pinpoint
Telescopes (HW look at

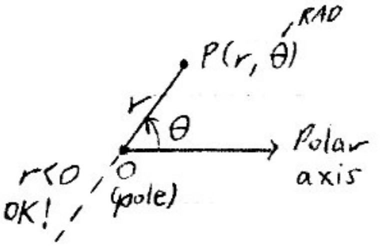
⑥ Applics.
Radar
Property: $|d_1 - d_2| = \text{const.}$
receives signal earlier; where would source be?
Reflecting property

James Bern.
intro PCs in 1691
but Newton
may have used 1st
Grid 361: 1st suggested
by N (≈1671)

10.8: POLAR COORDS (PCs)

Ⓐ PCs

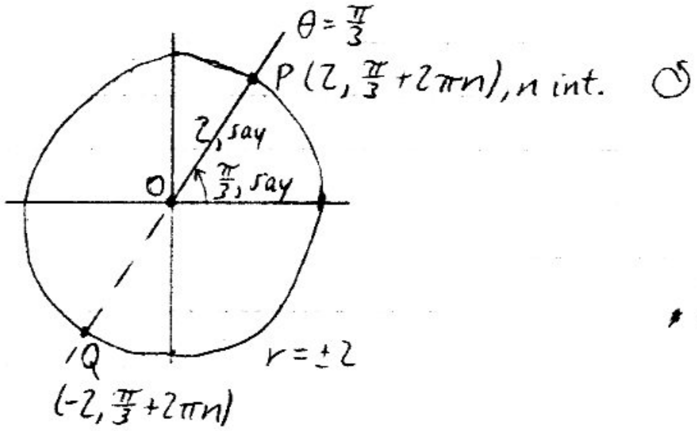
I've buried a treasure chest. You can ask me 2 ?s.
Rect/Cart. → Michael Golton fix.

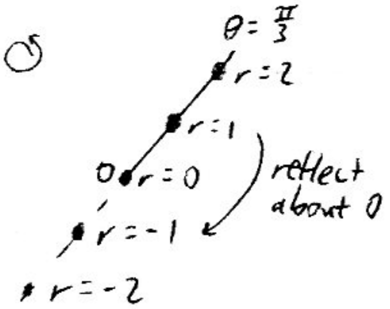


Pole O: $(0, \theta)$ — <u>any</u> angle

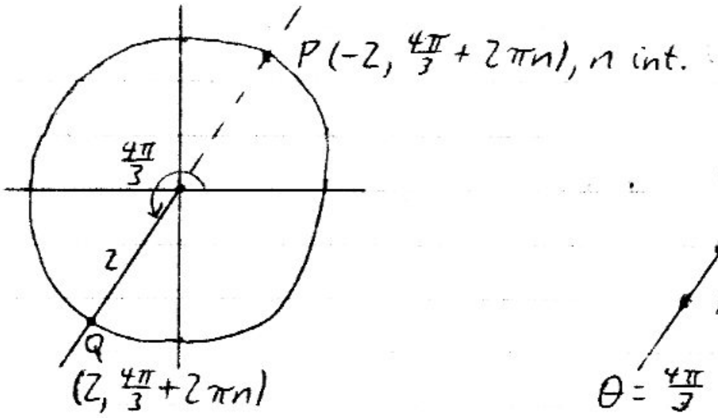
P has ∞ many PC reps.

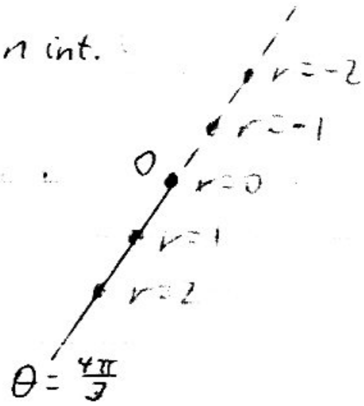
Hands! graph $r=2$? what pts have r-coord $=2$? What figure do they form.





or





Ⓑ Graph of a Polar Eq.

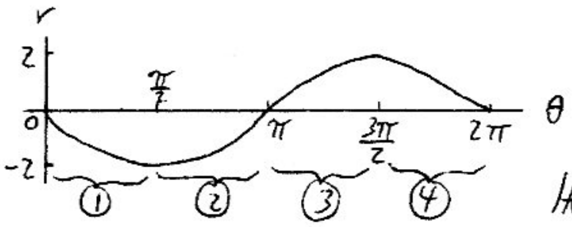
consists of all pts. (r, θ)
satisfy eq.

Usual form: r or $r^2 = f(\theta)$

Ex $\quad r = -2\sin\theta$

Stewart 673 "in Cartesian coords"

Graph r vs. θ as Cartesian/rectangular coords.



Be careful about calling these Q ($r<0$, later: ch θ))

r: $0 \to -2 \to 0 \to 2 \to 0$

Here, "sectors" corresp. to quadrants, but be careful!

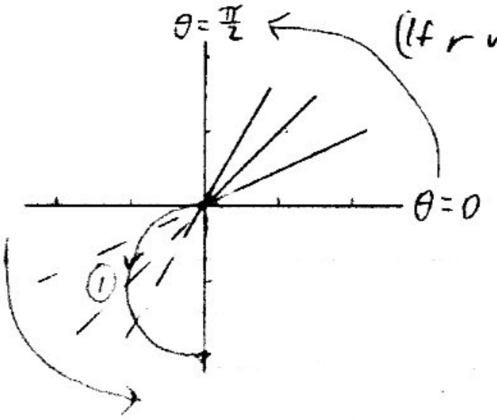
or Table:

θ	r
0	$-2\sin(0)=0$
⋮	

If $r>0$, we'd be going thru QI

Do ①

(If r were >0)

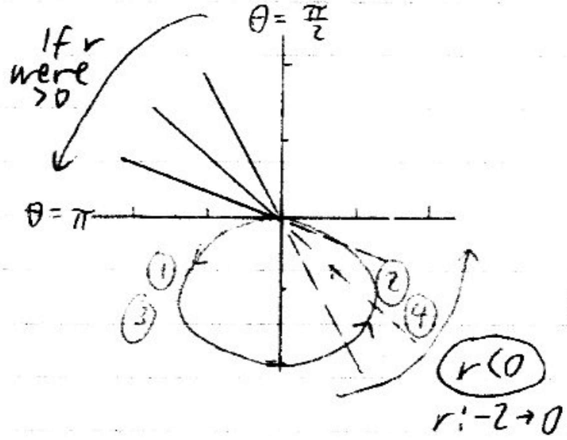


Trust me: it's a half-circle
Recognize basic forms

$r<0$

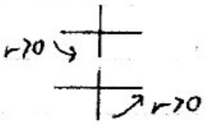
$r: 0 \to -2$

Do (2)



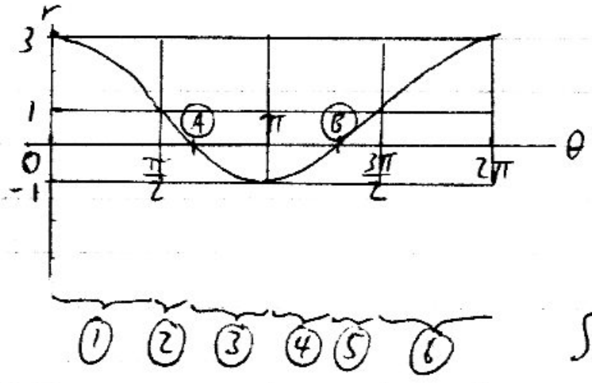
Circle — We'll show how you can prove this later.

(3) retraces (1)
(4) (2)



Ex $r = 1 + 2\cos\theta$

Note:
$-2 \le 2\cos\theta \le 2$
$-1 \le 1 + 2\cos\theta \le 3$



$r: 3 \longrightarrow 1 \rightarrow 0 \rightarrow -1 \rightarrow 0 \rightarrow 1 \longrightarrow 3$

Sectors divided by:
- quadrants
- ↗ vs. ↘
- + vs. −

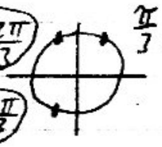
Find A, B

$$0 = 1 + 2\cos\theta$$
$$\cos\theta = -\frac{1}{2}$$

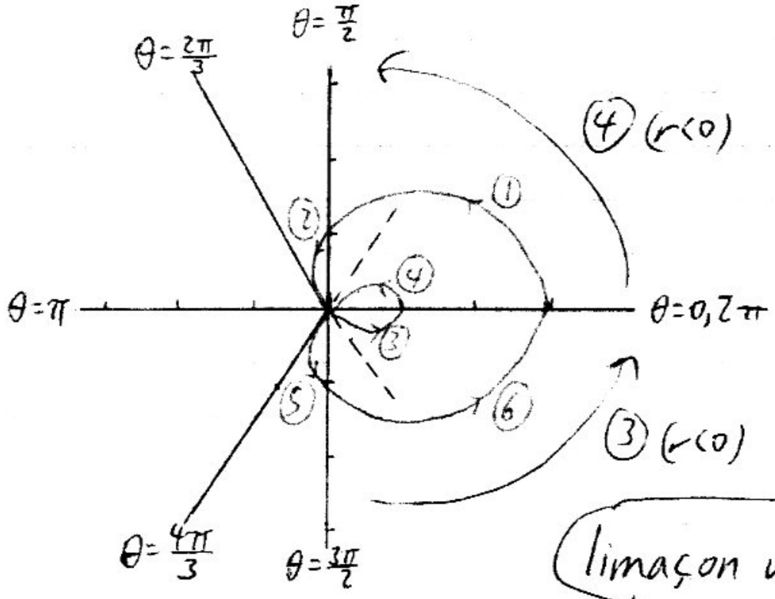
Discuss'

$A = \frac{2\pi}{3}$

$B = \frac{4\pi}{3}$



$\frac{\pi}{3}: \cos = \frac{1}{2}$



limaçon w/a loop

snail

Lee-muh-SOH"
Webster: locus set θ
fr snail
escargot?
'prepared snail

Stewart 677
$r=1+c\sin\theta$
Me
$r=1+a\cos\theta$
$a=1$ cardioid
$a\neq 0,1$: limaçon
$0<a<1$: no loop
$\frac{1}{2}<a<1$: dimple

$0<a\leq\frac{1}{2}$
no dimple
oval (a term 677: "convex")
$a\to 0$, $\to$ circle
$r=1+\cos\theta$

$r=1+\cos\theta$

cardioid — "heart" (butt?)

The evolution of $r=1+a\cos\theta$

$r=1+0.7\cos\theta$

dimpled limaçon

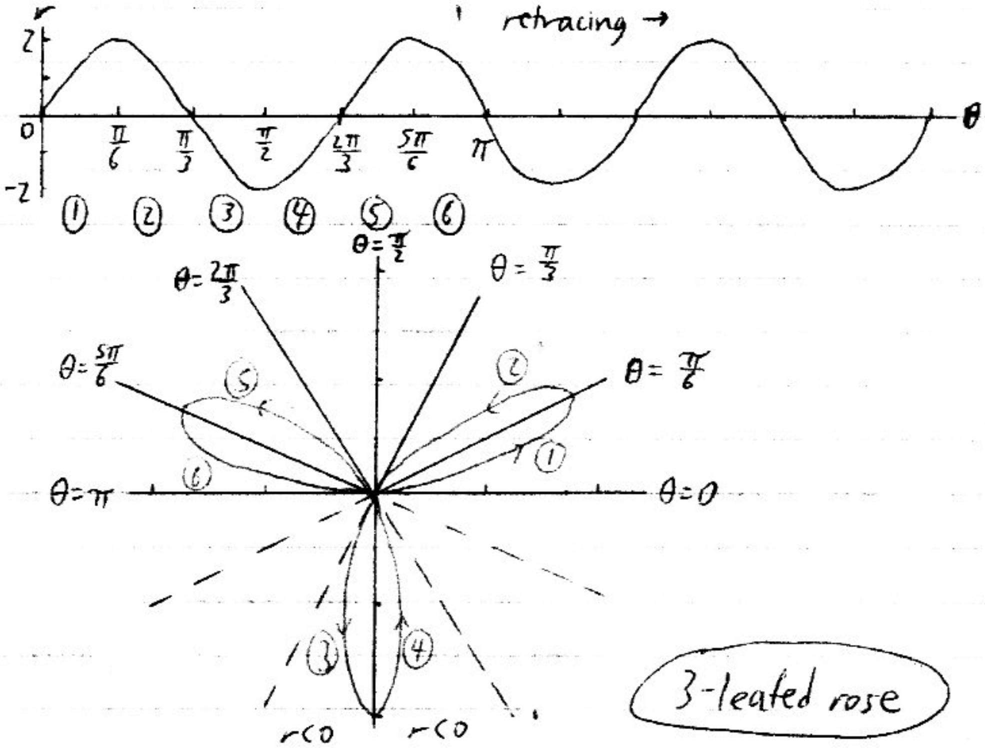
$r=1+0.5\cos\theta$ (loses dimple)

loses dimple
still limaçons

$r=1$

Circle!

Ex $r = 2\sin(3\theta)$



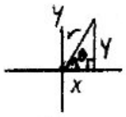
3-leafed rose

$$r = a \begin{matrix}\sin \\ \cos\end{matrix} (n\theta)$$

$a \neq 0$ — if $n = 3, 5, 7, \ldots \Rightarrow n$ leaves

if $n = 2, 4, 6, \ldots \Rightarrow 2n$ leaves

Ⓒ Polar Eq. $\Leftrightarrow$ Rect. Eq.



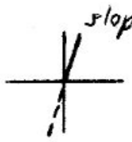
x (Polar)

$$\boxed{\begin{array}{l} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x},\ x \neq 0 \end{array}}$$ ← Watch quadrant?

$$\cos\theta = \frac{x}{r} \Rightarrow \quad \boxed{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ \text{even if } r<0 \end{array}}$$
$$\sin\theta = \frac{y}{r} \Rightarrow$$

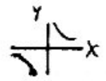
Ex Find a polar eq. w/same graph as $y = 6x$

$$\frac{y}{x} = 6 \quad \text{Also, } (0,0)$$
$$\tan\theta = 6$$
$$\boxed{\theta = \tan^{-1} 6}$$



Ex Find a rect. eq. w/same graph as $r^2 \sin(2\theta) = 4$, and graph it.

$$r^2(2\sin\theta\cos\theta) = 4$$
$$2\underbrace{(r\sin\theta)}_{=y}\underbrace{(r\cos\theta)}_{=x} = 4$$
$$2xy = 4$$
$$xy = 2 \quad \text{or} \quad y = \frac{2}{x}$$



hyp.

Ex (again) $r = -2\sin\theta$

$$r^2 = -2r\sin\theta$$
$$x^2 + y^2 = -2y$$
$$x^2 + y^2 + 2y = 0$$
$$x^2 + (y^2 + 2y + 1) = 1$$
$$x^2 + (y+1)^2 = 1$$

✓ when $r=0$ OK?

Circle w/center: $(0,-1)$
radius $= 1$

OK✓