CHAPTER 10:

Conic Sections and Polar Coordinates

- Conic sections are cross sections of a cone.

- The four types of conic sections are circles, parabolas, ellipses, and hyperbolas.

- Parametric equations can be used to describe oriented curves.

- Polar coordinates draw upon ideas we saw with vectors in Chapter 6. They are an alternative to Cartesian coordinates for a plane.

- Polar curves in the plane can be determined by equations in the polar coordinates r and θ.

SECTION 10.1: CONIC SECTIONS

LEARNING OBJECTIVES

• Know the four types of conic sections.

PART A: DISCUSSION

• Conic sections are cross sections of a cone.

• There are four types of conic sections:

§ Circles
(which we studied in Section 0.13)

§ Parabolas
(which we studied in Section 2.1; we will see more in Section 10.2)

§ Ellipses
(in Section 10.3)

§ Hyperbolas
(in Section 10.4).

• Circles are not technically ellipses, but it is often helpful to compare circles and ellipses.

• A video on conic sections may be found here:

https://www.youtube.com/watch?v=HO2zAU3Eppo

Some corrections to the video:

• The definition of **directrix** in the video is not commonly accepted.

• A true cone technically has no "base"; it consists of two **nappes** of infinite surface area.

SECTION 10.2: PARABOLAS

LEARNING OBJECTIVES

- Know the locus (geometric) definition of a parabola.
- Locate the focus and directrix of a parabola.

PART A: DISCUSSION

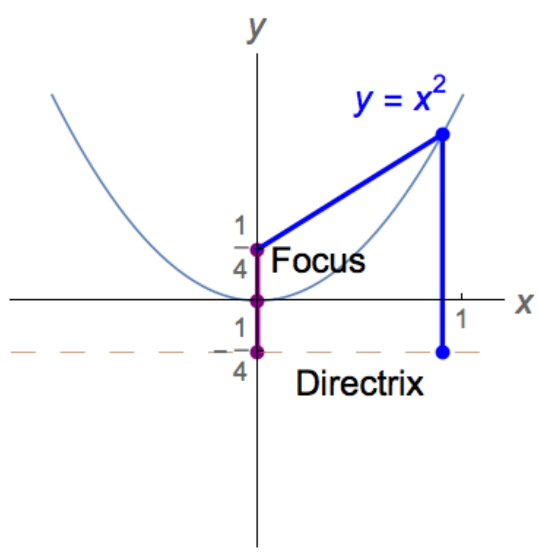
- In Section 2.1, we discussed **parabolas**. The key point that we identified on the parabola was the **vertex** (the turning point).

- Geometrically, a **parabola** is determined by a point (the **focus**) and a line (the **directrix**), both in a plane.

PART B: THE LOCUS (GEOMETRIC) DEFINITION OF A PARABOLA

A **parabola** consists of all points that are **equidistant** between a fixed point (called the focus) and a fixed line (called the directrix).

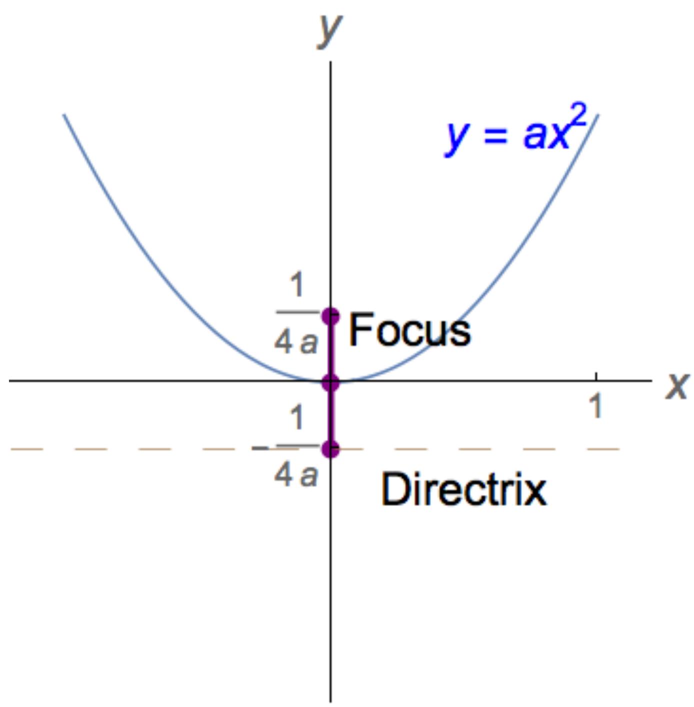
- The **vertex** of the parabola given by $y = x^2$, which is at $(0, 0)$, has **distance** $\frac{1}{4}$ from both the **focus** and the **directrix**.



- The **vertex** is the point on the parabola that is **closest to the focus**.

- The **vertex** is also the point on the parabola that is **closest to the directrix**.

• More generally, the **vertex** of the parabola $y = ax^2\ (a > 0)$ has **distance** $\frac{1}{4a}$ from both the **focus** and the **directrix**.



• A video on how to **construct a parabola** is here; the **directrix** would be somewhere in the middle of the page, not at the bottom:

https://www.youtube.com/watch?v=M9g7jXrMyeo

PART C: REFLECTIVE PROPERTIES

• All of the conic sections have **reflective properties** involving their **foci** (the plural of focus) that are useful in physics.

• These videos demonstrate the **reflective properties** of parabolas:

https://www.youtube.com/shorts/DK1piIfMA34

• Rays emanating from the focus reflect off the parabola in **parallel rays**.

• If a parabola is revolved about its axis, the resulting three-dimensional surface is called a **paraboloid**. Where are paraboloids found on a car?

SECTION 10.3: ELLIPSES

LEARNING OBJECTIVES

- Know the locus (geometric) definition of an ellipse.
- Obtain standard form for the equation of an ellipse.
- Locate the center, vertices, co-vertices, and foci of an ellipse.
- Find the eccentricity of an ellipse.

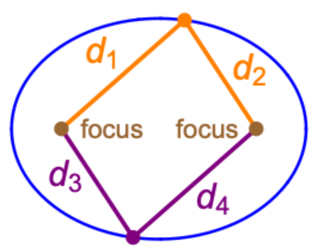
PART A: DISCUSSION

- Geometrically, an **ellipse** is determined by two points (the **foci**) in a plane.
- The **center**, **foci**, and **vertices** of an ellipse all lie on the **major axis** of the ellipse.
- The **eccentricity** of an ellipse helps describe its shape.
- The **standard form** of the equation of an ellipse can be used to find the center, foci, vertices, co-vertices, and eccentricity of the ellipse.

PART B: THE LOCUS (GEOMETRIC) DEFINITION OF AN ELLIPSE

An ellipse consists of all points whose **sum of distances** from two fixed points (called the **foci**, the plural of **focus**) in a plane stays constant.

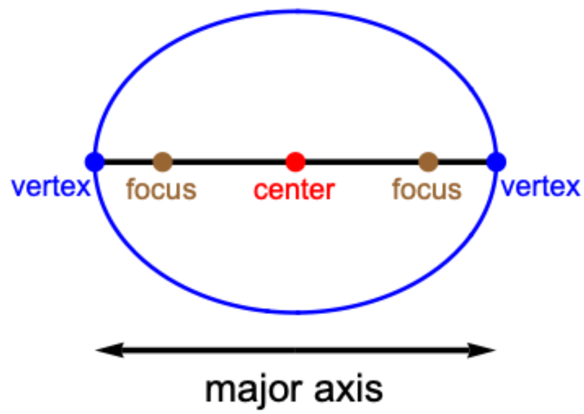
Below, $d_1 + d_2 = d_3 + d_4$:



- Imagine that the **two ends** of a string are being pinned down at the **two foci**. A marker stretches out the string until it is taut. If the marker is placed at the **orange point** in the above figure, then the length of the string is given by $d_1 + d_2$. Similarly, if the marker is placed at the **purple point**, then the length of the string is given by $d_3 + d_4$.
- A video on how to **construct an ellipse** is here:

https://www.youtube.com/watch?v=7UD8hOs-vaI

PART C: TERMINOLOGY



• An ellipse has a **center**, but, unlike a circle, an ellipse does not have a constant radius or diameter.

• The major axis of an ellipse can be thought of as the "longest diameter" in the ellipse.

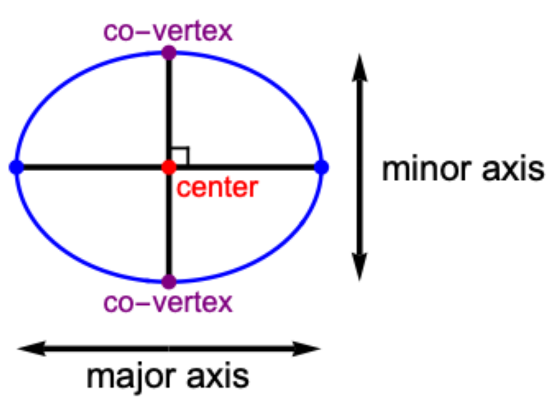
§ Terms such as "major axis" may refer to **line segments** or their **lengths**.

§ The **vertices** are the **endpoints** of the major axis.

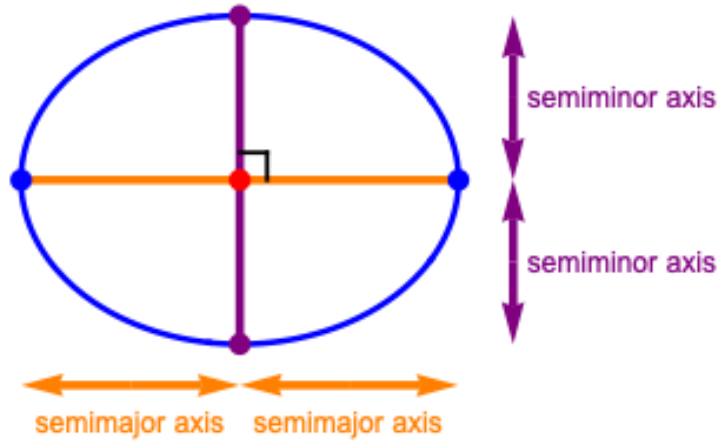
§ The **center** is the **midpoint** of the major axis.

§ The two **foci** lie between the center and the two vertices; **all five points lie on the major axis**, and they are **symmetric about the center**.

• The minor axis can be thought of as the "shortest diameter" that passes through the **center**. It is perpendicular to the major axis.



§ The **endpoints** of the minor axis are sometimes called the **co-vertices** of the ellipse.



• The <u>semimajor axis</u> of an ellipse can be thought of as the "longest radius" in the ellipse.

§ Think: "**Half** of the major axis."

§ Geometrically, there are really **two** semimajor axes, each extending from the **center** to each **vertex**. **Together, they make up the major axis.**

• The <u>semiminor axis</u> can be thought of as the "shortest radius" in the ellipse.

§ Think: "**Half** of the minor axis."

§ Geometrically, there are really **two** semiminor axes, each extending from the **center** to each **co-vertex**. **Together, they make up the minor axis.**

PART D: NOTATION; HORIZONTAL and VERTICAL ELLIPSES

Standard Notation for Lengths and Distances in an Ellipse

a = semimajor axis
$2a$ = major axis

b = semiminor axis
$2b$ = minor axis

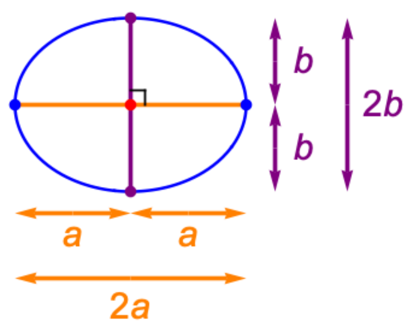
c = distance from the center to each focus

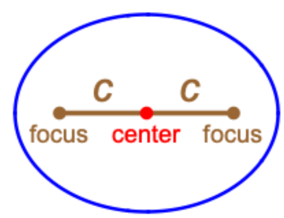
We assume that $a, b, c > 0$.

Below, the **blue** points are the **vertices**. The **purple** points are the **co-vertices**.

Horizontal ("x-long") Ellipse

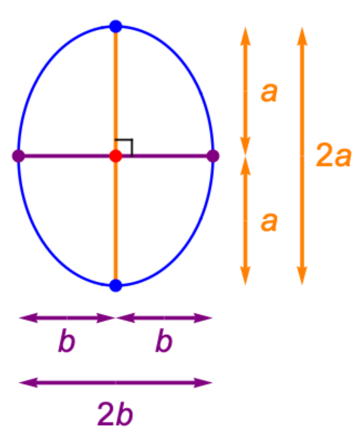
A **horizontally** elongated ellipse has a **horizontal** major axis.

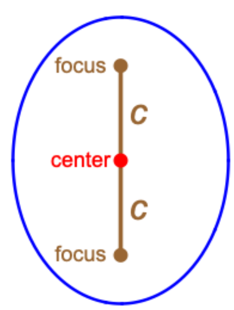




Vertical ("y-long") Ellipse

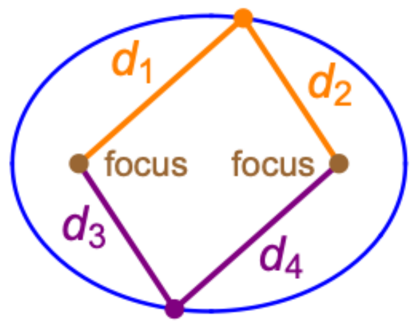
A **vertically** elongated ellipse has a **vertical** major axis.





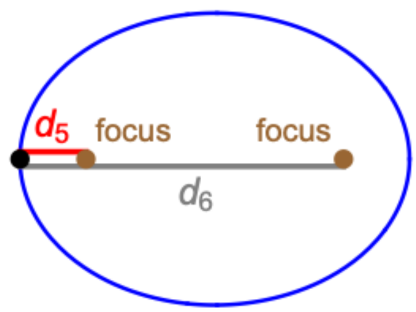
PART E: MAJOR AXIS AS STRING LENGTH

In Part B, we saw that $d_1 + d_2$ and $d_3 + d_4$ could be thought of as the total length of a string used to construct the ellipse:

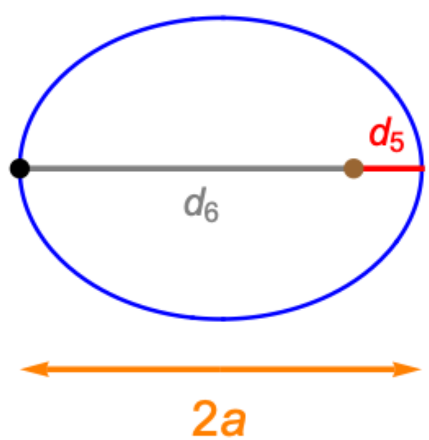


In fact, in a pleasant surprise, it turns out that they both equal the **major axis**, $2a$. **The length of the string is 2*a*.**

Imagine that our marker is at the black point indicated below.



The **length of the string** is $d_5 + d_6$. Imagine cutting off and moving the **red** piece of the string over to the **right** end of the **gray** piece of the string.



Observe that the **length of the string**, $d_5 + d_6 = 2a$, the **major axis**.

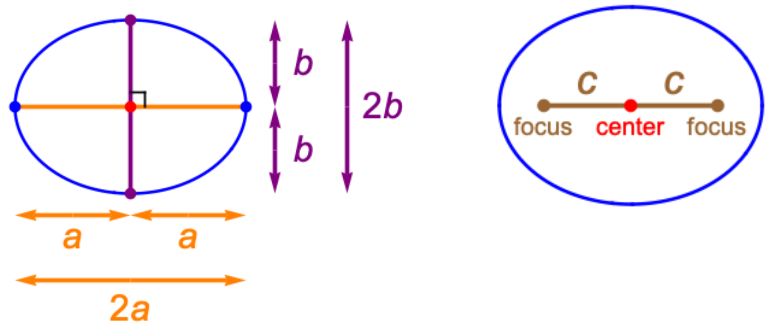
PART F: FORMULA FOR LOCATING FOCI

Formula for Locating the Foci of an Ellipse (Finding c)

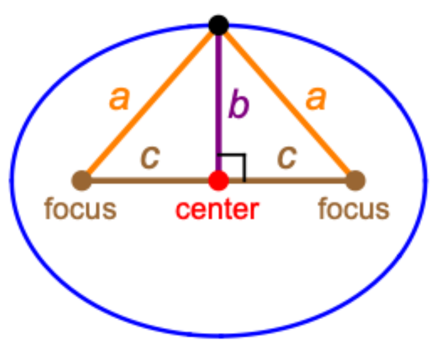
$$c^2 = a^2 - b^2$$

WARNING 1: Signs. Although the **Pythagorean Theorem** is behind this formula, we have a **minus sign** $(-)$ on the right-hand side. We are going to apply the Pythagorean Theorem to obtain the formula $a^2 = b^2 + c^2$. (See also Footnote 1.)

Remember the setup for a horizontal ellipse:



Imagine that our marker is at the **black** point at the **top** of the ellipse.
In Part E, we showed that **the total length of the orange string is 2*a***.
By **symmetry**, the marker separates the string into two pieces, each of length a.



By the **Pythagorean Theorem**,

$$a^2 = b^2 + c^2$$
$$a^2 - b^2 = c^2$$
$$c^2 = a^2 - b^2$$

(Remember that $a, b, c > 0$.) Note that $c^2 < a^2$ and $c < a$. This makes sense, since the **foci** are closer to the **center** than the **vertices** are.

By definition of the semimajor and semiminor axes, we also have: $b < a$.

PART G: ECCENTRICITY

The eccentricity of an ellipse is denoted by e. It is a measure of how **elongated** the ellipse is.

$$e = \frac{c}{a}$$

Note: This e has nothing to do with the number e (about 2.718) that we saw in Chapter 3.

For an ellipse, $0 < e < 1$.

For an ellipse, we already know (see Part F):

$$0 < c < a$$

Divide all three parts by a:

$$\frac{0}{a} < \frac{c}{a} < \frac{a}{a}$$

$$0 < e < 1$$

Play with this applet on ellipses:
https://www.mathopenref.com/ellipsesemiaxes.html

Other conic sections:

We still use:

$$e = \frac{c}{a}$$

• A **circle** has $e = 0$. Imagine the foci of ellipses converging towards the center.

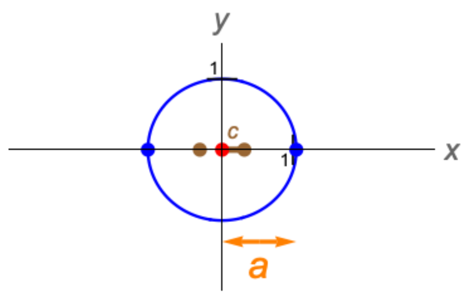
• A **parabola** has $e = 1$. This comes from the general definition of eccentricity. Let P be a point on a conic section, and let F be the focus that is closest to that point. The distance between P and F divided by the perpendicular distance between P and a fixed line (the directrix) gives the eccentricity, and it remains constant for the points on the conic section. For a parabola, this is 1. See: https://www.mathsisfun.com/geometry/eccentricity.html

• **A hyperbola** has $e > 1$. See Section 10.4.

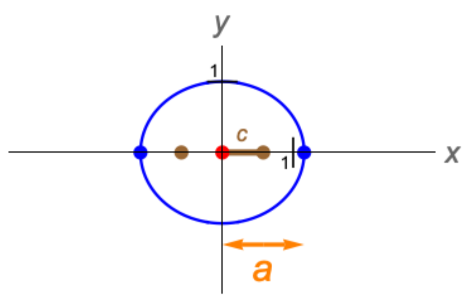
Comparing Eccentricities of Ellipses:

The **higher** e is, the more **elongated** the ellipse is.

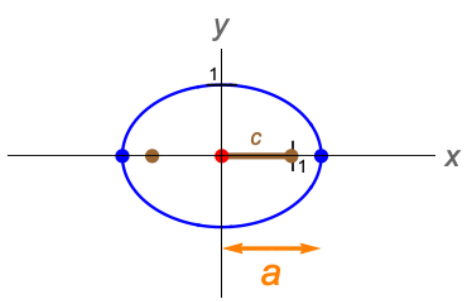
For the ellipse below, $e = 0.3$.
c is 30% of a. That is, the distance between each **focus** and the **center** is 30% of the distance between each **vertex** and the **center**.



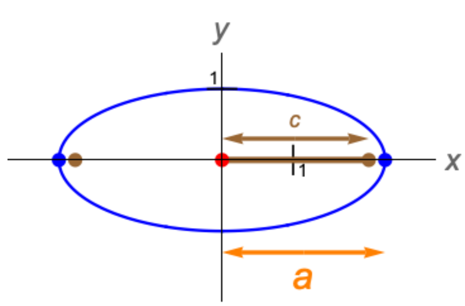
For the ellipse below, $e = 0.5$. (c is 50% of a.)



For the ellipse below, $e = 0.7$. (c is 70% of a.)



For the ellipse below, $e = 0.9$. (c is 90% of a.)



If you're curious, see Footnote 2 on the equations of these ellipses.

PART H: STANDARD FORM FOR THE EQUATION OF AN ELLIPSE

Standard Form for the Equation of an Ellipse (with Center at (0, 0))

For a **horizontal** ellipse, it is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } a > b.$$

For a **vertical** ellipse, it is:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \text{ where } a > b.$$

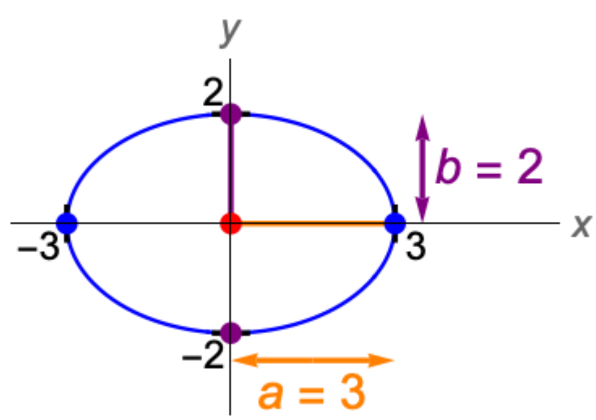
• If the larger denominator is below x^2, then we have a **horizontal (“*x*-long”)** ellipse.

• If the larger denominator is below y^2, then we have a **vertical (“*y*-long”)** ellipse.

Note: If $a = b$, then we would have a **circle** of radius a.

Example 1 (Standard Form; Horizontal Ellipse with Center at (0,0))

The graph of $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is below.



• The **larger** denominator is 9, so $a^2 = 9$ and $a = 3$.

• Because the **larger** denominator is below the x^2, the ellipse is **horizontal (“*x*-long”)**.

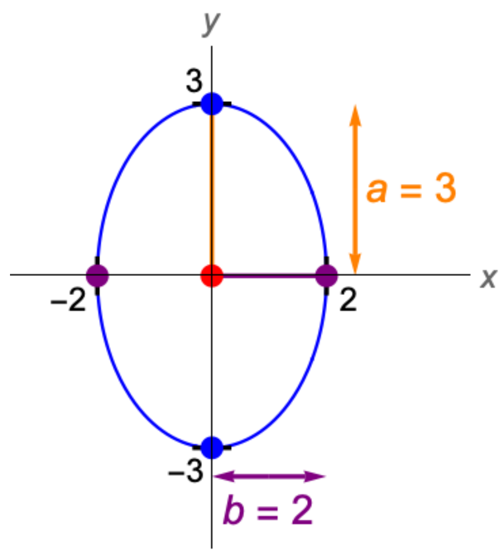
• The **smaller** denominator is 4, so $b^2 = 4$ and $b = 2$.

• If we set $y = 0$, we see that the ***x*-intercepts** are at $(-3, 0)$ and $(3, 0)$.
This makes sense, since we have a horizontal ellipse with center at $(0, 0)$ and $a = 3$.

• If we set $x = 0$, we see that the ***y*-intercepts** are at $(0, -2)$ and $(0, 2)$.
This makes sense, since we have a horizontal ellipse with center at $(0, 0)$ and $b = 2$. §

Example 2 (Standard Form; Vertical Ellipse with Center at (0, 0))

The graph of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is below.



• The **larger** denominator is 9, so $a^2 = 9$ and $a = 3$.

• Because the **larger** denominator is below the y^2, the ellipse is **vertical (“*y*-long”)**.

• The **smaller** denominator is 4, so $b^2 = 4$ and $b = 2$.

• If we set $y = 0$, we see that the ***x*-intercepts** are at $(-2, 0)$ and $(2, 0)$.
This makes sense, since we have a vertical ellipse with center at $(0, 0)$ and $b = 2$.

• If we set $x = 0$, we see that the ***y*-intercepts** are at $(0, -3)$ and $(0, 3)$.
This makes sense, since we have a vertical ellipse with center at $(0, 0)$ and $a = 3$. §

To **translate** the ellipse so that the **new center** is at (h, k), use Section 1.4, Part F on **translations through coordinate shifts**. See also Section 0.13 on **circles**.

- We replace x with $(x - h)$.
- We replace y with $(y - k)$.

<u>Standard Form for the Equation of an Ellipse (with Center at (h, k))</u>

For a **horizontal** ellipse, it is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \text{ where } a > b.$$

For a **vertical** ellipse, it is:

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1, \text{ where } a > b.$$

PART I: AN EXAMPLE

Example 3 (Finding Standard Form; Ellipse with Center at (h, k))

Consider the ellipse given by $9x^2 + 4y^2 - 54x + 40y + 37 = 0$.
Find the standard form of the equation of this ellipse.

§ Solution

$$9x^2 + 4y^2 - 54x + 40y + 37 = 0$$

- **Group** the terms containing x, and **group** the terms containing y.
- **Isolate constant terms** on the right-hand side**.**

$$\left(9x^2 - 54x\right) + \left(4y^2 + 40y\right) = -37$$

- **Factor the leading coefficients** out of each group. (Fractions may result.)

$$9\left(x^2 - 6x\right) + 4\left(y^2 + 10y\right) = -37$$

- **Complete the square (CTS)** within each pair of parentheses.
- **Balance** the equation.

For instance, to CTS within the "new x group," $\left(x^2 - 6x\right)$:
- •• Take the **coefficient of** x, -6.
- •• **Halve** it, resulting in -3.
- •• **Square** the result. We obtain $+9$.
- •• **Add** the $+9$ to the group.

We obtain the **Perfect Square Trinomial (PST)**: $\left(x^2 - 6x + 9\right)$.

WARNING 2: Remember to **balance** the equation properly.
Do not ignore the impact of the **leading coefficients**, 9 and 4.

For instance, when $+9$ is inserted in the "new x group," then 9(9), or 81, is being added to the left-hand side. This must be **balanced** out by adding 9(9), or 81, to the right-hand side.

$$9\left(x^2 - 6x + 9\right) + 4\left(y^2 + 10y + 25\right) = -37 + 9(9) + 4(25)$$

• **Factor the PSTs** as squares of binomials. For instance, observe that -3 is half of -6, the coefficient of x in $\left(x^2 - 6x + 9\right)$.

$$9(x-3)^2 + 4(y+5)^2 = 144$$

• **Divide** both sides of the equation by the **constant term** on the right-hand side. This ensures that "1" is isolated on the right-hand side.

$$\frac{9(x-3)^2}{144} + \frac{4(y+5)^2}{144} = 1$$

$$\frac{(x-3)^2}{16} + \frac{(y+5)^2}{36} = 1$$

Note: We were fortunate here because 144 was divisible by both 9 and 4. If we had had, say,

$$\frac{9(x-3)^2}{143} + \frac{4(y+5)^2}{143} = 1$$

then we should observe that, when we multiply by a nonzero number, we are **dividing by its reciprocal**. For instance, when we multiply $(x-3)^2$ by $\dfrac{9}{143}$, we are dividing $(x-3)^2$ by $\dfrac{143}{9}$.

$$\frac{(x-3)^2}{\frac{143}{9}} + \frac{(y+5)^2}{\frac{143}{4}} = 1$$

§

Example 4 (Finding the Center; Revisiting Example 3)

Find the **center** of the ellipse given by $\dfrac{(x-3)^2}{16}+\dfrac{(y+5)^2}{36}=1$ in Example 3.

§ Solution

The **center** of the ellipse is at (3, −5).

WARNING 3: Beware of signs. The same trick we used for **circles** works here. Ask, "What makes the left-hand side equal to 0?"

Observe that in the standard form $\dfrac{(x-h)^2}{b^2}+\dfrac{(y-k)^2}{a^2}=1$,
$(x-3)$ is of the form $(x-h)$, where $h=3$, and
$(y+5)$, or $(y-(-5))$, is of the form $(y-k)$, where $k=-5$. §

Example 5 (Finding the Vertices; Revisiting Example 3)

Find the **vertices** of the ellipse given by $\dfrac{(x-3)^2}{16}+\dfrac{(y+5)^2}{36}=1$ in Example 3.

§ Solution

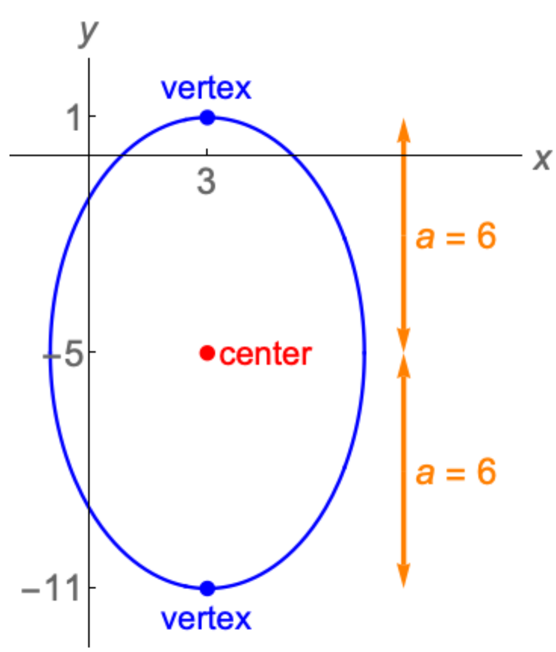
• The **larger** denominator is 36, so $a^2 = 36$ and $a = 6$.

• Because the **larger** denominator is below the $(y+5)^2$, the ellipse is **vertical ("*y*-long")**, and the **major and semimajor axes are vertical**.

• Therefore, one **vertex** lies 6 units directly **above** the **center**, $(3, -5)$; the other **vertex** lies 6 units directly **below** the **center**.

• For a **vertical** ellipse, the **vertices** and the **center** all share the same ***x*-coordinates**. The ***y*-coordinate** of the **center** (-5) is to be shifted 6 units up and 6 units down to get the ***y*-coordinates** of the **vertices**. This may remind you of the PCAPIA / Frame Method for graphing trigonometric functions in Chapter 4.

• The **vertices** are at $(3, 1)$ and $(3, -11)$.



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Example 6 (Finding the Co-vertices; Revisiting Example 3)

Find the **co-vertices** of the ellipse given by $\frac{(x-3)^2}{16}+\frac{(y+5)^2}{36}=1$ in Example 3.

§ Solution

• The **smaller** denominator is 16, so $b^2=16$ and $b=4$.

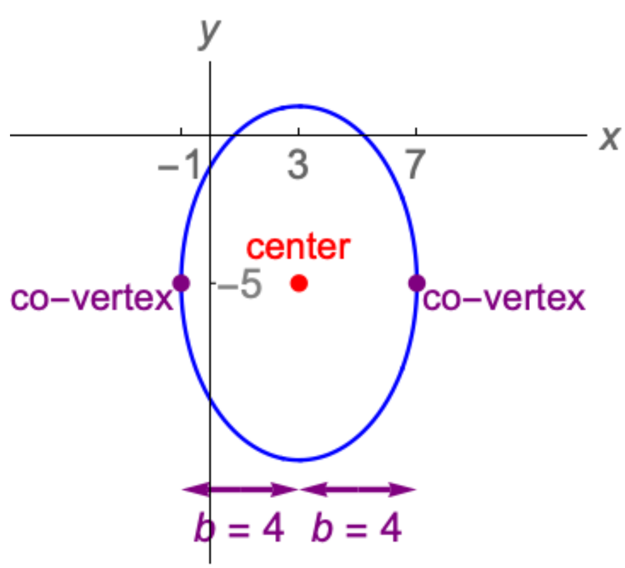
• We know that the ellipse is **vertical ("*y*-long")**, so the **minor and semiminor axes are horizontal**.

WARNING 4: Which coordinate do we shift? To move from the **center** to the **co-vertices**, we move along the **minor axis**. This is different from moving from the **center** to the **vertices** or to the **foci**; these involve moving along the **major axis**.

• Therefore, one **co-vertex** lies 4 units directly **to the right** of the **center**, (3, −5); the **other** lies 4 units directly **to the left** of the **center**.

• For a **vertical** ellipse, the **co-vertices** and the **center** all share the same ***y*-coordinates**. The ***x*-coordinate** of the **center** (3) is to be increased by 4 units and decreased by 4 units to get the ***x*-coordinates** of the **co-vertices**.

• The **co-vertices** are at (7, −5) and (−1, −5).



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Example 7 (Finding the Foci; Revisiting Example 3)

Find the **foci** of the ellipse given by $\dfrac{(x-3)^2}{16} + \dfrac{(y+5)^2}{36} = 1$ in Example 3.

§ Solution

• Find c, the distance between the **center** and the **foci**.

$$c^2 = a^2 - b^2 = 36 - 16 = 20 \Rightarrow$$
$$c = \sqrt{20} = \sqrt{4 \cdot 5} = 2\sqrt{5}$$

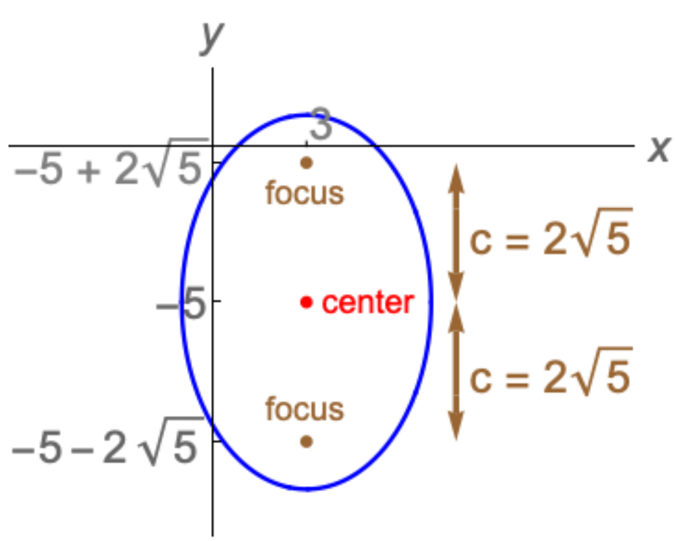
• Locating the **foci** is similar to locating the **vertices**, except that we move c units away from the **center** instead of a units.

• We know that the ellipse is **vertical ("*y*-long")**, so the **major axis is vertical**.

• The **center**, **vertices**, and **foci** all share the same ***x*-coordinate**.

• The ***y*-coordinate** of the **center** (-5) is to be shifted $2\sqrt{5}$ units up and $2\sqrt{5}$ units down to get the ***y*-coordinates** of the **foci**.

• The **foci** are at $\left(3,\ -5+2\sqrt{5}\right)$ and $\left(3,\ -5-2\sqrt{5}\right)$.
These are approximately at: (3, −0.53) and (3, −9.47).



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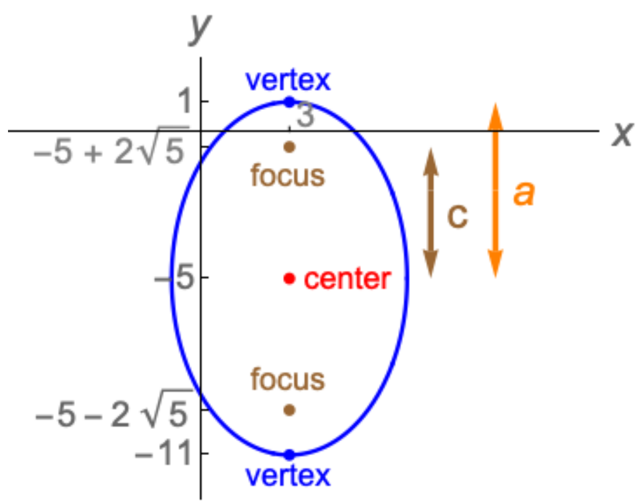
Example 8 (Finding the Eccentricity; Revisiting Example 3)

Find the **eccentricity** of the ellipse given by $\dfrac{(x-3)^2}{16}+\dfrac{(y+5)^2}{36}=1$ in Example 3.

§ Solution

• The eccentricity $e=\dfrac{c}{a}=\dfrac{2\sqrt{5}}{6}=\dfrac{\sqrt{5}}{3}\approx 0.745$

• Note that c is about 75% of a in the figure below.



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PART J: REFLECTIVE PROPERTIES and APPLICATIONS

These videos demonstrate the **reflective properties** of ellipses and their applications:

• **Animation:**
https://www.youtube.com/shorts/5V-TL1WWtl4

• **An elliptical pool table:**
https://www.youtube.com/watch?v=4KHCuXN2F3I

• **National Statuary Hall at the U.S. Capitol.** John Quincy Adams made good use of the reflective property when eavesdropping on rivals!
https://www.youtube.com/watch?v=FX6rUU_74kk

• **Lithotripsy.** This common medical procedure uses shock waves (sound waves) to break up kidney stones without harming surrounding tissue.
http://mathcentral.uregina.ca/beyond/articles/Lithotripsy/lithotripsy1.html

PART K: KEPLER'S LAWS

• **Kepler's First Law** of planetary motion states that each planet in our solar system travels in an **elliptical orbit** about the **Sun**, which is at one **focus**. (**Earth's orbit** has an eccentricity of about 0.0167.)

• **Kepler's Second Law** states that, if we imagine a line connecting the Sun and a planet, then this line sweeps out **equal areas in equal time intervals**.

• **Kepler's Third Law** relates the orbital period of a planet (its "year") to the semimajor axis of the orbit.

For more:

- From NASA: https://solarsystem.nasa.gov/resources/310/orbits-and-keplers-laws/
- Wikipedia: https://en.wikipedia.org/wiki/Kepler%27s_laws_of_planetary_motion
- Animation: https://www.youtube.com/watch?v=Dvoe8Ib5D1o

Comets also travel in elliptical orbits, although some may travel in parabolic or hyperbolic orbits.

- **Halley's comet** has an eccentricity of about 0.967, and it takes about 76 years to complete one orbit around the sun.
- From NASA: https://nssdc.gsfc.nasa.gov/planetary/factsheet/cometfact.html

FOOTNOTES

1. **Deriving the equation of a basic ellipse; Formula for locating foci.**
   See https://mathworld.wolfram.com/Ellipse.html for all this and more.
   Thanks to Pat McKeague for the approach in Part F.

2. **Equations of the ellipses in Part G.**

   For a horizontal ellipse centered at the origin, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

   Set $b = 1$ so that the co-vertices are at $(0, 1)$ and $(0, -1)$. We then have: $\frac{x^2}{a^2} + y^2 = 1$.

   For a desired value of e, what should a^2 be? We need to express a^2 in terms of e.

   $e = \frac{c}{a}$, so $a = \frac{c}{e}$ and $a^2 = \frac{c^2}{e^2}$.

   Since $c^2 = a^2 - b^2$ and $b = 1$, we have $c^2 = a^2 - 1$.

   Then, $a^2 = \frac{c^2}{e^2}$ becomes $a^2 = \frac{a^2 - 1}{e^2}$.

   Solving for a^2, we obtain: $a^2 = \frac{1}{1 - e^2}$.

   The equation $\frac{x^2}{a^2} + y^2 = 1$ can then be written as:

   $\frac{x^2}{\frac{1}{1-e^2}} + y^2 = 1$ (in standard form), or

   $\left(1 - e^2\right)x^2 + y^2 = 1$.

SECTION 10.4: HYPERBOLAS

LEARNING OBJECTIVES

- Know the locus (geometric) definition of a hyperbola.
- Obtain standard form for the equation of a hyperbola.
- Locate the center, vertices, and foci of a hyperbola.
- Find the equations of the asymptotes of a hyperbola.
- Find the eccentricity of a hyperbola.

PART A: DISCUSSION

- Geometrically, a **hyperbola** is also determined by two points (the **foci**) in a plane.

- A **hyperbola** consists of two **branches**, and they approach the two **asymptotes** of the hyperbola.

- The **terminology** used for a hyperbola is **not as consistent** across sources as for an ellipse.

- The **eccentricity** of a hyperbola helps describe its shape.

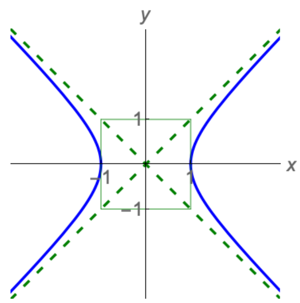
- The **standard form** of the equation of a hyperbola can be used to find the center, foci, vertices, eccentricity, as well as the **asymptotes** of the hyperbola.

PART B: FAMOUS HYPERBOLAS

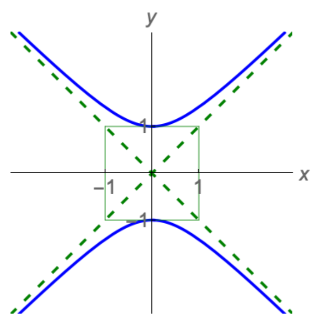
• The graph of $x^2 - y^2 = 1$ is the most famous **horizontal** hyperbola.
Its branches **open horizontally**.

We will call the **green square** the central box of the hyperbola.
(For other hyperbolas, this will be a **rectangle**.)

The **dashed green asymptotes** are drawn through the **diagonals** of this box.

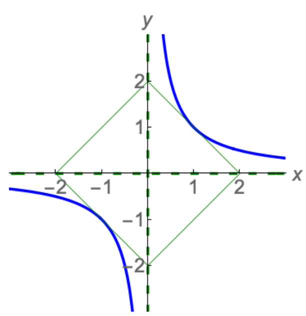


• The graph of $y^2 - x^2 = 1$ is the most famous **vertical** hyperbola.
Its branches **open vertically**.



• The graph of $y = \frac{1}{x}$, or $xy = 1$, which we discussed in Section 1.3, is a **rotated** hyperbola. The **asymptotes** lie on the coordinate axes.

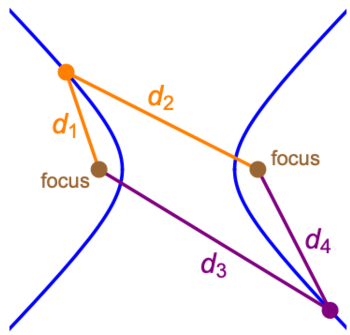
Note: It is obtained by rotating the hyperbola given by $x^2 - y^2 = 2$, or $\frac{x^2}{2} - \frac{y^2}{2} = 1$ (in standard form), counterclockwise about the origin by $45°$.



PART C: THE LOCUS (GEOMETRIC) DEFINITION OF A HYPERBOLA

A hyperbola consists of all points whose **absolute value of the difference of distances** from two fixed points (the **foci**) in a plane stays constant.

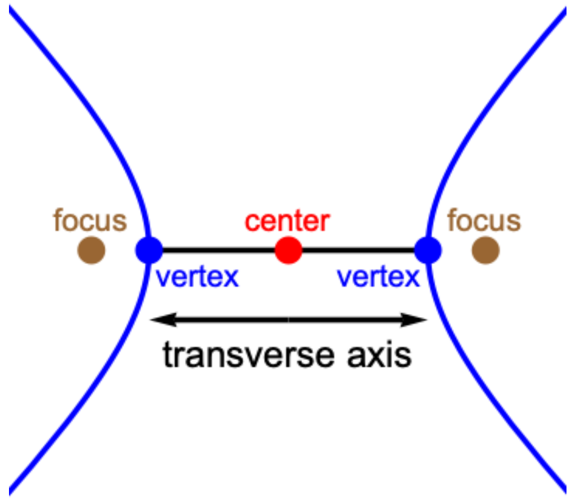
Below, $|d_1 - d_2| = |d_3 - d_4|$:



• A video on how to **construct a hyperbola** is here:
https://www.youtube.com/watch?v=H8NwiZFS7B0

• Note: As for ellipses, we will denote the distance between the vertices as $2a$. Although $|d_1 - d_2| = |d_3 - d_4| = 2a$, this observation is not as helpful for hyperbolas as the observation $d_1 + d_2 = d_3 + d_4 = 2a$ was for ellipses.

PART D: TERMINOLOGY



• The **vertices** of a hyperbola are the two points on the two branches that are closest to each other.

• We will call the line segment connecting the two vertices (as endpoints) the transverse axis. It is the "shortest distance" between the two branches of the hyperbola.

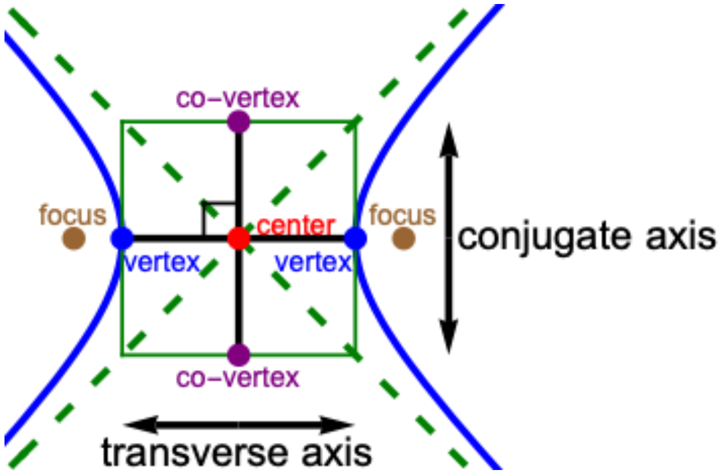
> **WARNING 1: Terminology for hyperbolas is inconsistent across sources.** Some sources say that the transverse axis is an **infinite line**, not just a line segment. Some sources use the term **major axis** for either the line or the line segment that we will call the transverse axis. One drawback to using the term **major axis** here is that, for a **hyperbola**, the major axis might be **shorter** than the minor axis, which can be confusing.

• The **center** of a hyperbola is the midpoint of the transverse axis.

• As for an **ellipse**, the **vertices** and the **foci** are **symmetric** about the **center**. All those points lie on the same line, the line that contains the transverse axis.

WARNING 2:

• Unlike for an ellipse, for a **hyperbola**, the **foci** are **further** from the **center** than the **vertices** are.



• The two **branches** of the hyperbola approach two **asymptotes**.

• The **central box** of the hyperbola is the rectangle or square with the same **center** as the hyperbola and with **diagonals** lying on the **asymptotes**.

• The **vertices** lie on the **central box**; they are the **midpoints** of opposing sides of the box.

• The conjugate axis of the hyperbola is the line segment with these properties:

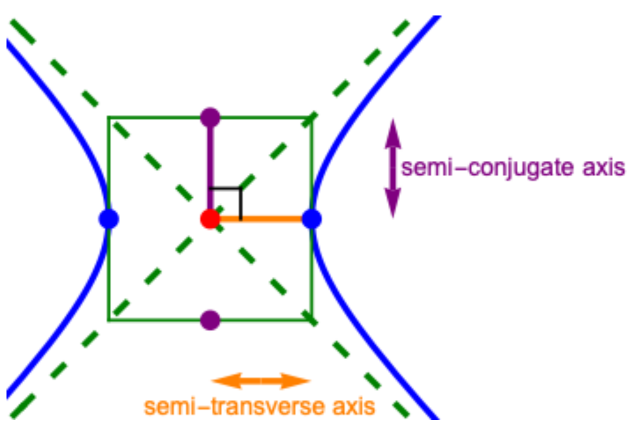
•• It is **perpendicular** to the transverse axis.

•• It passes through the **center** of the hyperbola (and of the central box).

•• Its **endpoints** (sometimes called the **co-vertices**) lie on the **central box**; they are the **midpoints** of opposing sides of the box.

WARNING 3: The co-vertices do not lie on the hyperbola itself.

• The semi-transverse axis is **half** of the transverse axis.
• The semi-conjugate axis is **half** of the conjugate axis.



PART E: NOTATION; HORIZONTAL and VERTICAL HYPERBOLAS

<u>Standard Notation for Lengths and Distances Related to a Hyperbola</u>

a = semi-transverse axis
$2a$ = transverse axis

b = semi-conjugate axis
$2b$ = conjugate axis

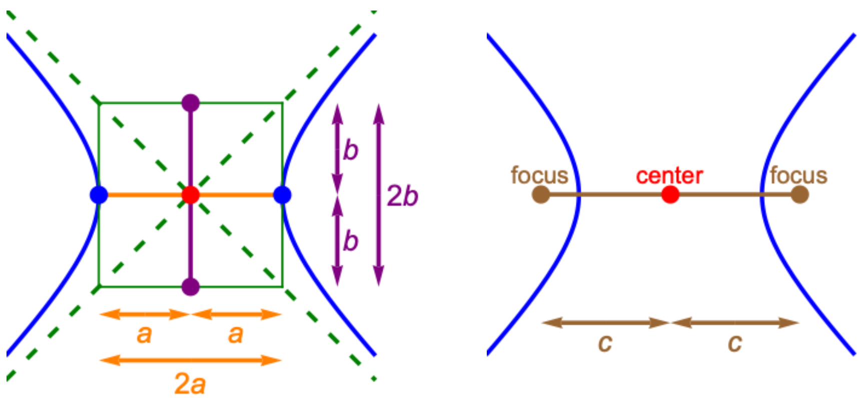
c = distance from the center to each focus

We assume that $a, b, c > 0$.

Below, the **blue** points are the **vertices**. The **purple** points are the **co-vertices**.

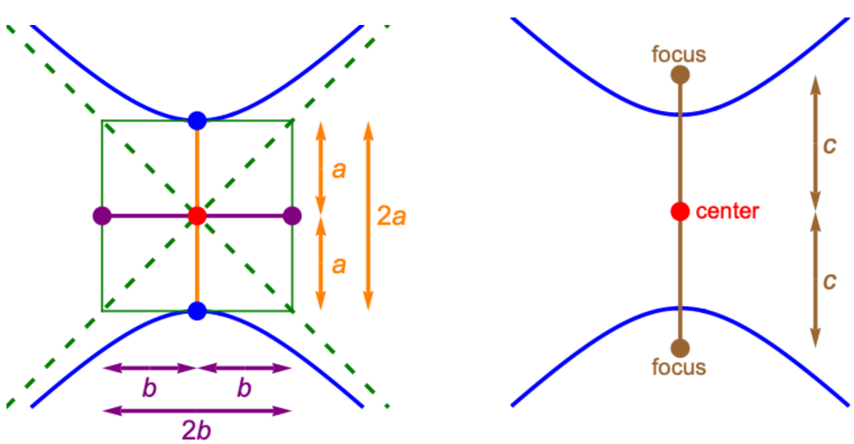
<u>Horizontal Hyperbola</u>

A **horizontal** hyperbola opens **horizontally** and has a **horizontal** transverse axis.



<u>Vertical Hyperbola</u>

A **vertical** hyperbola opens **vertically** has a **vertical** transverse axis.



PART F: FORMULA FOR LOCATING FOCI

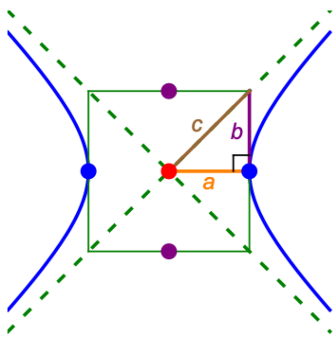
Formula for Locating the Foci of a Hyperbola (Finding c)

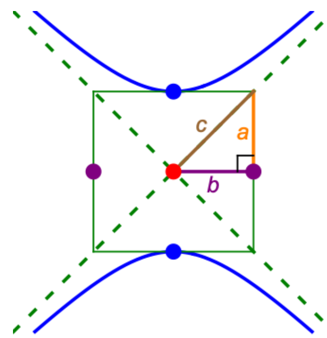
$$c^2 = a^2 + b^2$$

WARNING 4: Signs. Unlike the formula for ellipses, this looks like the usual form of the **Pythagorean Theorem**. Ironically, it is harder to explain! (See Footnote 1.)

(Remember that a, b, $c > 0$.) Note that $c^2 > a^2$ and $c > a$. This makes sense, since the **foci** are further away from the **center** than the **vertices** are.

c turns out to be **half** the length of a **diagonal** of the **central box**.





PART G: ECCENTRICITY

The **eccentricity** of a hyperbola is denoted by e and is again given by:

$$e = \frac{c}{a}$$

For a hyperbola, $e > 1$.

For a hyperbola, we already know (see Part F):

$$c > a$$

Divide both parts by a:

$$\frac{c}{a} > \frac{a}{a}$$

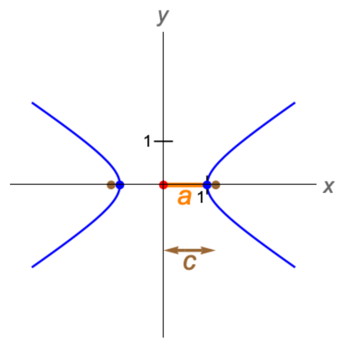
$$e > 1$$

Comparing Eccentricities of Hyperbolas:

The **higher** e is, the more the branches are **pulled away from** the line containing the **transverse axis**.

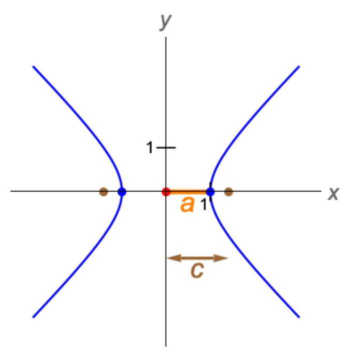
For the hyperbola below, $e = 1.2$.
c is 120% of a. That is, the distance between each **focus** and the **center** is 120% of the distance between each **vertex** and the **center**.

$e = 1.2$



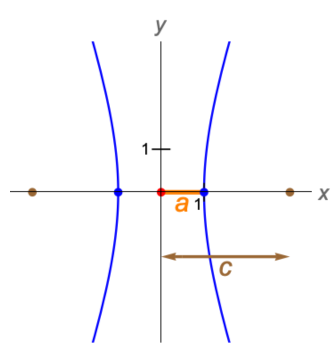
For the hyperbola below, $e = \sqrt{2} \approx 1.4$. (c is about 140% of a.)
In fact, this is the graph of $x^2 - y^2 = 1$.

$e \approx 1.4$



For the hyperbola below, $e = 3$. (c is three times a.)

$e = 3$



If you're curious, see Footnote 2 on the equations of these hyperbolas.

PART H: STANDARD FORM FOR THE EQUATION OF A HYPERBOLA; EQUATIONS OF ASYMPTOTES

Standard Form for the Equation of a Hyperbola (with Center at (0, 0)); Equations of Asymptotes

For a **horizontal** hyperbola, it is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and the **asymptotes** are given by:

$$y = \pm \frac{b}{a} x$$

For a **vertical** hyperbola, it is:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

and the **asymptotes** are given by:

$$y = \pm \frac{a}{b} x$$

WARNING 5: The patterns are **different** from the ones for **ellipses**.

- If x^2 is in the term on the **left** of the minus sign $(-)$, then we have a **horizontal** hyperbola.
- If y^2 is in the term on the **left** of the minus sign $(-)$, then we have a **vertical** hyperbola.
- In both cases, a^2 is the denominator of the fraction to the **left** of the minus sign $(-)$.

Notes on Asymptotes:

• The "$\pm$" symbol can be ambiguous. In this section, "$y = \pm \frac{b}{a}x$" means that $y = \frac{b}{a}x$ gives one asymptote and $y = -\frac{b}{a}x$ gives the other.

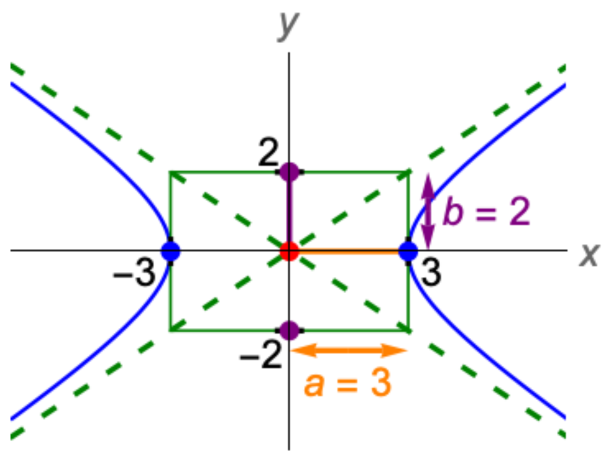
• Remember that $\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\triangle y}{\triangle x}$.

For the **horizontal** hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, think of b as the "y-partner" and a as the "x-partner." The **asymptotes** have slopes $\pm \frac{y-\text{partner}}{x-\text{partner}} = \pm \frac{b}{a}$.

For the **vertical** hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, think of a as the "y-partner" and b as the "x-partner." The **asymptotes** have slopes $\pm \frac{y-\text{partner}}{x-\text{partner}} = \pm \frac{a}{b}$.

Example 1 (Standard Form; Horizontal Hyperbola with Center at (0,0))

The graph of $\frac{x^2}{9} - \frac{y^2}{4} = 1$ is below.



• x^2 is in the term on the **left** of the minus sign $(-)$, so we have a **horizontal** hyperbola.

• 9 is the denominator of the fraction to the **left** of the minus sign $(-)$, so $a^2 = 9$ and $a = 3$.

• If we set $y = 0$, we see that the ***x*-intercepts** are at $(-3, 0)$ and $(3, 0)$. This makes sense, since we have a **horizontal** hyperbola with center at $(0, 0)$ and $a = 3$.

• If we set $x = 0$, we see that there are **no *y*-intercepts**.

• 4 is the denominator of the fraction to the **right** of the minus sign $(-)$, so $b^2 = 4$ and $b = 2$. This helps us find the **co-vertices** and **central box** of the hyperbola.

• For this **horizontal** hyperbola, the **asymptotes** are given by:

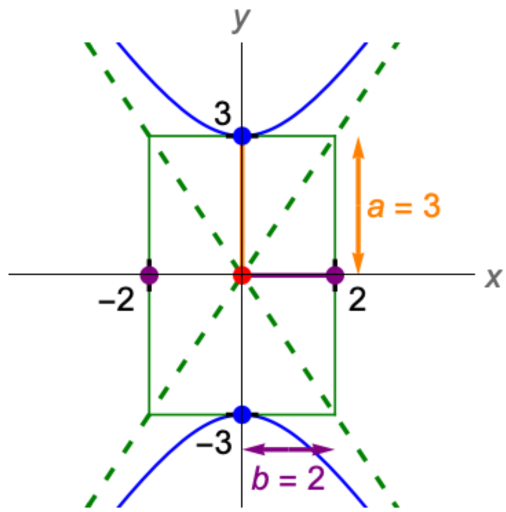
$$y = \pm \frac{b}{a} x$$

$$y = \pm \frac{2}{3} x$$

For the slopes, think: $\pm \frac{y - \text{partner}}{x - \text{partner}} = \pm \frac{b}{a} = \pm \frac{2}{3}$. §

Example 2 (Standard Form; Vertical Hyperbola with Center at (0, 0))

The graph of $\frac{y^2}{9} - \frac{x^2}{4} = 1$ is below.



• y^2 is in the term on the **left** of the minus sign $(-)$, so we have a **vertical** hyperbola.

• 9 is the denominator of the fraction to the **left** of the minus sign $(-)$, so $a^2 = 9$ and $a = 3$.

• If we set $y = 0$, we see that there are **no *x*-intercepts**.

• If we set $x = 0$, we see that the ***y*-intercepts** are at $(0, -3)$ and $(0, 3)$. This makes sense, since we have a **vertical** hyperbola with center at $(0, 0)$ and $a = 3$.

• 4 is the denominator of the fraction to the **right** of the minus sign $(-)$, so $b^2 = 4$ and $b = 2$. This helps us find the **co-vertices** and **central box** of the hyperbola.

• For this **vertical** hyperbola, the **asymptotes** are given by:

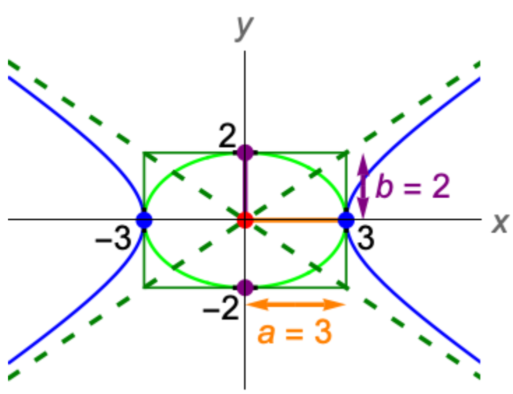
$$y = \pm \frac{a}{b} x$$

$$y = \pm \frac{3}{2} x$$

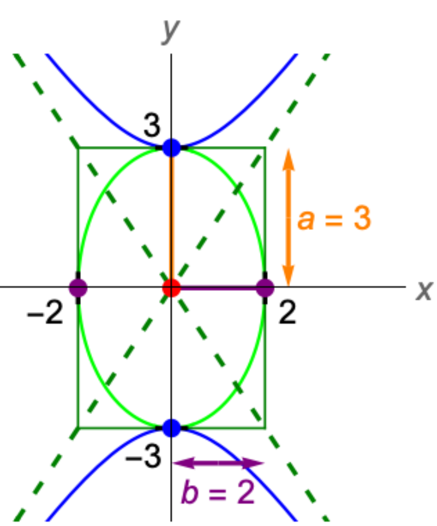
For the slopes, think: $\pm \frac{y - \text{partner}}{x - \text{partner}} = \pm \frac{a}{b} = \pm \frac{3}{2}$. §

In the figure below, compare the graphs of $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$ (a **hyperbola**) and $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ (an **ellipse**).

- Since $a > b$ for the hyperbola, the **hyperbola** and the **ellipse** have the same **vertices** and the same **co-vertices**.
- If $a = b$, then we get a **circle** instead of an ellipse.
- If $a < b$, then the **vertices** of the hyperbola are the **co-vertices** of the ellipse.



In the figure below, we compare the graphs of $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$ (a **hyperbola**) and $\dfrac{y^2}{9} + \dfrac{x^2}{4} = 1$ (an **ellipse**).



To **translate** the hyperbola so that the **new center** is at (h, k), use Section 1.4, Part F on **translations through coordinate shifts**. See also Section 0.13 on **circles** and Section 10.3 on **ellipses**.

- We replace x with $(x-h)$.
- We replace y with $(y-k)$.

Standard Form for the Equation of a Hyperbola (with Center at (h, k)); Equations of Asymptotes

For a **horizontal** hyperbola, it is:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

and the **asymptotes** are given by:

$$y-k=\pm\frac{b}{a}(x-h)$$

$$y=k\pm\frac{b}{a}(x-h)$$

For a **vertical** hyperbola, it is:

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

and the **asymptotes** are given by:

$$y-k=\pm\frac{a}{b}(x-h)$$

$$y=k\pm\frac{a}{b}(x-h)$$

PART I: AN EXAMPLE

Example 3 (Finding Standard Form; Hyperbola with Center at (h, k))

Consider the hyperbola given by $y^2 - 4x^2 - 12y - 16x + 16 = 0$.
Find the standard form of the equation of this hyperbola.

§ Solution

$$y^2 - 4x^2 - 12y - 16x + 16 = 0$$

- **Group** the terms containing *x*, and **group** the terms containing *y*.
- Since y^2 has a **positive** leading coefficient, let's start with the "*y* group."
- **Isolate constant terms** on the right-hand side**.**

$$\left(y^2 - 12y\right) + \left(-4x^2 - 16x\right) = -16$$

WARNING 6: Signs. Start by separating the two groups with a plus sign $(+)$. Students often make mistakes when trying to factor out a (-1) mentally.

- **Factor the leading coefficients** out of each group. (Fractions may result.) We are fortunate that the leading coefficient for the "*y* group" here is 1.

$$\left(y^2 - 12y\right) - 4\left(x^2 + 4x\right) = -16$$

WARNING 7: Signs. Watch your signs when factoring!

- **Complete the square (CTS)** within each pair of parentheses.
- **Balance** the equation.

For instance, to CTS within the "new *y* group," $\left(y^2 - 12y\right)$:

- •• Take the **coefficient of *y***, -12.
- •• **Halve** it, resulting in -6.
- •• **Square** the result. We obtain $+36$.
- •• **Add** the $+36$ to the group.

We obtain the **Perfect Square Trinomial (PST)**:
$\left(y^2 - 12y + 36\right)$.

WARNING 8: Remember to **balance** the equation properly. Consider the **leading coefficients** (1 and -4 here).

For instance, when $+4$ is inserted in the "new x group," then $-4(4)$, or -16, is being added to the left-hand side. This must be **balanced** out by adding $-4(4)$, or -16, to the right-hand side.

$$\left(y^2 - 12y + 36\right) - 4\left(x^2 + 4x + 4\right) = -16 + 36 - 4(4)$$

• **Factor the PSTs** as squares of binomials. For instance, observe that -6 is half of -12, the coefficient of y in $\left(y^2 - 12y + 36\right)$.

$$(y-6)^2 - 4(x+2)^2 = 4$$

• **Divide** both sides of the equation by the **constant term** on the right-hand side. This ensures that "1" is isolated on the right-hand side.

$$\frac{(y-6)^2}{4} - \frac{4(x+2)^2}{4} = 1$$

$$\frac{(y-6)^2}{4} - \frac{(x+2)^2}{1} = 1, \text{ or}$$

$$\frac{(y-6)^2}{4} - (x+2)^2 = 1$$

Note 1: Sometimes, we should observe that, when we multiply by a nonzero number, we are **dividing by its reciprocal**. For instance, if we had had $(y-6)^2 - 4(x+2)^2 = 5$ (instead of 4):

$$(y-6)^2 - 4(x+2)^2 = 5$$

$$\frac{(y-6)^2}{5} - \frac{4(x+2)^2}{5} = 1$$

When we multiply $(x+2)^2$ by $\frac{4}{5}$, we are dividing $(x+2)^2$ by $\frac{5}{4}$.

$$\frac{(y-6)^2}{5} - \frac{(x+2)^2}{\frac{5}{4}} = 1$$

Note 2 / **WARNING 9: Signs.** If we had ended up with a **negative** number on the right-hand side, then we would need to do some rearranging to obtain standard form.

For instance, if we had had $(y-6)^2 - 4(x+2)^2 = -4$ (instead of 4):

$$(y-6)^2 - 4(x+2)^2 = -4$$

$$\frac{(y-6)^2}{-4} - \frac{4(x+2)^2}{-4} = 1$$

$$-\frac{(y-6)^2}{4} + \frac{(x+2)^2}{1} = 1$$

$$\frac{(x+2)^2}{1} - \frac{(y-6)^2}{4} = 1$$

§

Example 4 (Finding the Center; Revisiting Example 3)

Find the **center** of the hyperbola given by $\dfrac{(y-6)^2}{4} - \dfrac{(x+2)^2}{1} = 1$ in Example 3.

§ Solution

The **center** of the hyperbola is at $(-2, 6)$.

WARNING 10: Order of coordinates. The same trick we used for **circles** and **ellipses** works here. Ask, "What makes the left-hand side equal to 0?" Since this is a **vertical** hyperbola, **it is a common error to switch** the x- and y-coordinates and write $(6, -2)$.

Observe that in the standard form $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$, $(x+2)$, or $(x-(-2))$, is of the form $(x-h)$, where $h = -2$, and $(y-6)$ is of the form $(y-k)$, where $k = 6$. §

Example 5 (Finding the Vertices; Revisiting Example 3)

Find the **vertices** of the hyperbola given by $\dfrac{(y-6)^2}{4}-\dfrac{(x+2)^2}{1}=1$ in Example 3.

§ Solution

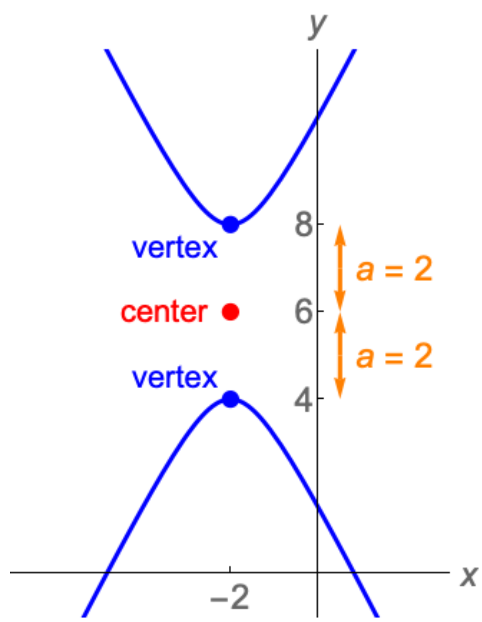
• $(y-6)^2$ is in the term on the **left** of the minus sign $(-)$, so the hyperbola is **vertical**, and the **transverse and semi-transverse axes are vertical**.

• 4 is the denominator of the fraction to the **left** of the minus sign $(-)$, so $a^2=4$ and $a=2$.

• Therefore, one **vertex** lies 2 units directly **above** the **center**, $(-2, 6)$; the other **vertex** lies 2 units directly **below** the **center**.

• For a **vertical** hyperbola, the **vertices** and the **center** all share the same ***x*-coordinates**. The ***y*-coordinate** of the **center** (6) is to be shifted 2 units up and 2 units down to get the ***y*-coordinates** of the **vertices**. This may remind you of the PCAPIA / Frame Method for graphing trigonometric functions in Chapter 4.

• The **vertices** are at $(-2, 8)$ and $(-2, 4)$.



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Example 6 (Finding the Co-vertices; Revisiting Example 3)

Find the **co-vertices** of the hyperbola given by $\frac{(y-6)^2}{4} - \frac{(x+2)^2}{1} = 1$ in Example 3.

§ Solution

• We know that the hyperbola is **vertical**, so the **conjugate and semi-conjugate axes are horizontal**.

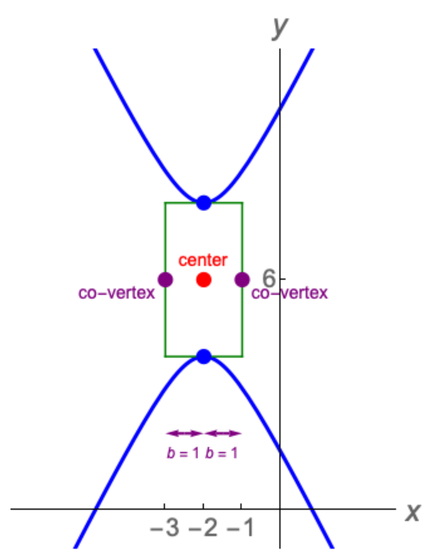
• 1 is the denominator of the fraction to the **right** of the minus sign $(-)$, so $b^2 = 1$ and $b = 1$. This helps us find the **co-vertices** and **central box** of the hyperbola.

WARNING 11: Which coordinate do we shift? To move from the **center** to the **co-vertices**, we move along the **conjugate axis**. This is different from moving from the **center** to the **vertices** or to the **foci**; these involve moving along the **transverse axis**.

• Therefore, one **co-vertex** lies 1 unit directly **to the right** of the **center**, $(-2, 6)$; the **other** lies 1 unit directly **to the left** of the **center**.

• For a **vertical** hyperbola, the **co-vertices** and the **center** all share the same ***y*-coordinates**. The ***x*-coordinate** of the **center** (-2) is to be increased by 1 unit and decreased by 1 unit to get the ***x*-coordinates** of the **co-vertices**.

• The **co-vertices** are at $(-1, 6)$ and $(-3, 6)$.



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Example 7 (Finding the Asymptotes; Revisiting Example 3)

Find the **asymptotes** of the hyperbola given by $\dfrac{(y-6)^2}{4} - \dfrac{(x+2)^2}{1} = 1$ in Example 3.

§ Solution

• For this **vertical** hyperbola, the **asymptotes** are given by:

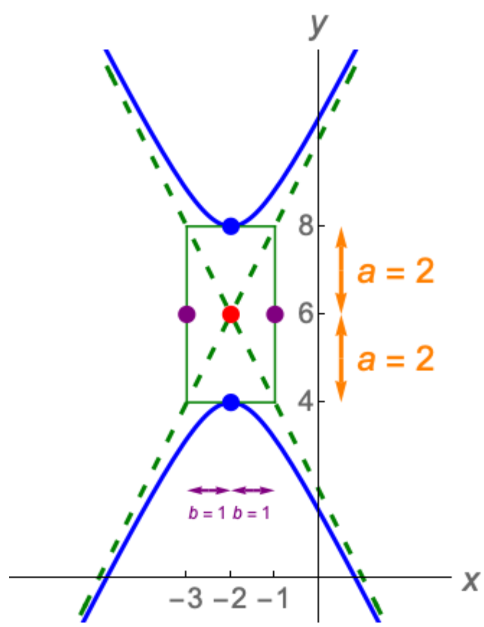
$$y - k = \pm \frac{a}{b}(x - h)$$

$$y - 6 = \pm \frac{2}{1}(x + 2)$$

$$y - 6 = \pm 2(x + 2), \text{ or}$$

$$y = 6 \pm 2(x + 2)$$

For the slopes, think: $\pm \dfrac{y-\text{partner}}{x-\text{partner}} = \pm \dfrac{a}{b} = \pm \dfrac{2}{1} = \pm 2.$



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Example 8 (Finding the Foci; Revisiting Example 3)

Find the **foci** of the hyperbola given by $\dfrac{(y-6)^2}{4} - \dfrac{(x+2)^2}{1} = 1$ in Example 3.

§ Solution

• Find c, the distance between the **center** and the **foci**.

$$c^2 = a^2 + b^2 = 4 + 1 = 5 \Rightarrow$$
$$c = \sqrt{5}$$

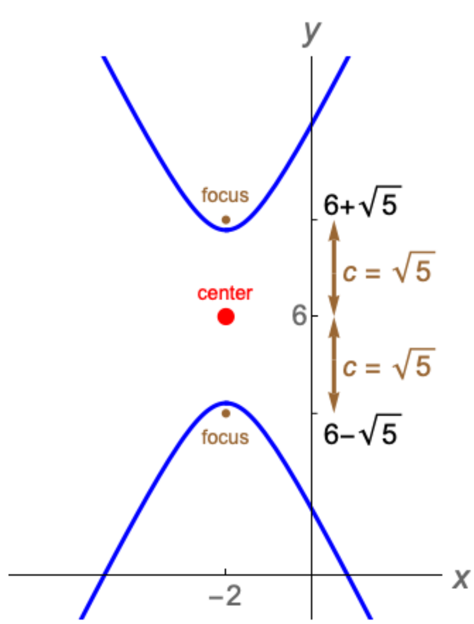
• Locating the **foci** is similar to locating the **vertices**, except that we move c units away from the **center** instead of a units.

• We know that the hyperbola is **vertical**, so the **transverse axis is vertical**.

• The **center**, **vertices**, and **foci** all share the same ***x*-coordinate**.

• The ***y*-coordinate** of the **center** (6) is to be shifted $\sqrt{5}$ units up and $\sqrt{5}$ units down to get the ***y*-coordinates** of the **foci**.

• The **foci** are at $\left(-2,\, 6+\sqrt{5}\right)$ and $\left(-2,\, 6-\sqrt{5}\right)$.
These are approximately at: $(-2, 8.24)$ and $(-2, 3.76)$.



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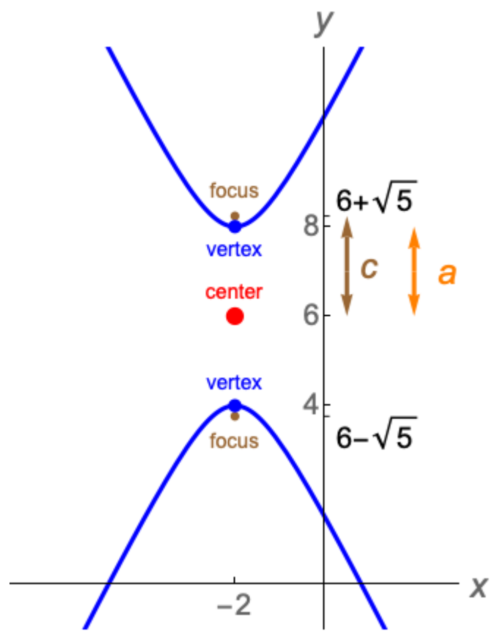
Example 9 (Finding the Eccentricity; Revisiting Example 3)

Find the **eccentricity** of the hyperbola given by $\frac{(y-6)^2}{4} - \frac{(x+2)^2}{1} = 1$ in Example 3.

§ Solution

• The eccentricity $e = \frac{c}{a} = \frac{\sqrt{5}}{2} \approx 1.12$

• Note that c is about 112% of a in the figure below.



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PART J: REFLECTIVE PROPERTIES

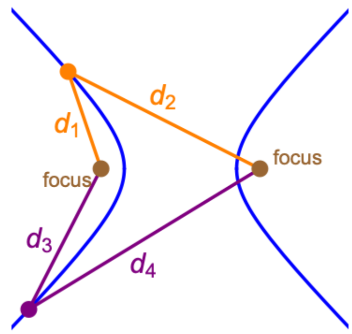
These videos demonstrate the **reflective properties** of hyperbolas:

• **Animations:**

https://www.youtube.com/shorts/D9eRk6LPBFc
https://www.youtube.com/shorts/VXUlPunpysk

PART K: APPLICATIONS (LORAN)

By the locus definition in Part C, for all pairs of points on the hyperbola, $\left|d_1 - d_2\right| = \left|d_3 - d_4\right|$:



• Let's say there are **two tracking stations** at the two **foci**. An explosion is heard, and the station on the left hears it five milliseconds before the station on the right.

• It could be determined that the explosion happened somewhere on the **left branch** of the above hyperbola.

• A **third tracking station** could be used to develop **another hyperbolic branch**, and the **intersection** of the two branches could be the location of the explosion.

• **LORAN (Long-Range Navigation)** was a land-based navigation system developed by M.I.T. during World War II. LORAN was based on these principles. (Source: *Encyclopedia Britannica*, https://www.britannica.com/technology/loran) See also: "Radio History: The Rise, Fall and Resurrection of LORAN."
https://blog.minicircuits.com/radio-history-the-birth-death-and-resurrection-of-loran/#:~:text=LORAN%20was%20declared%20obsolete%20in%202009%20and%20decommissioned%20in%202010

PART L: OTHER APPLICATIONS

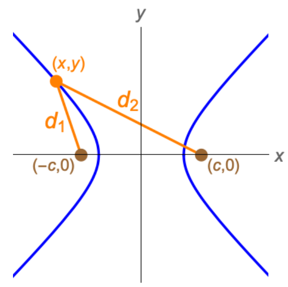
• Hyperbolic and parabolic mirrors are used in **telescopes** due to their reflective properties.

• Most **cooling towers** at nuclear power plants are in the shape of **hyperboloids**. A hyperboloid is a three-dimensional surface obtained by revolving a hyperbola about its conjugate axis (treated as an infinite line). See the Duke Energy communication at:

https://nuclear.duke-energy.com/2021/10/14/cooling-towers-what-are-they-and-how-do-they-work#:~:text=The%20shape%20of%20most%20cooling,laden%20air%20into%20the%20atmosphere

FOOTNOTES

1. **Deriving the equation of a basic hyperbola; Formula for locating foci.**
   See https://mathworld.wolfram.com/Hyperbola.html for this and more.
   See https://mathworld.wolfram.com/Ellipse.html for a similar analysis for ellipses.
   Consider the left branch of the hyperbola below; the right branch leads to the same equation.



$$d_2 - d_1 = 2a$$

$$\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = 2a$$

$$\sqrt{(x-c)^2+y^2} = 2a + \sqrt{(x+c)^2+y^2} \quad \Rightarrow$$

$$\left(\sqrt{(x-c)^2+y^2}\right)^2 = \left(2a + \sqrt{(x+c)^2+y^2}\right)^2$$

$$(x-c)^2 + y^2 = 4a^2 + 4a\sqrt{(x+c)^2+y^2} + (x+c)^2 + y^2$$

$$(x-c)^2 = 4a^2 + 4a\sqrt{(x+c)^2+y^2} + (x+c)^2$$

$$x^2 - 2cx + c^2 = 4a^2 + 4a\sqrt{(x+c)^2+y^2} + x^2 + 2cx + c^2$$

$$-2cx = 4a^2 + 4a\sqrt{(x+c)^2+y^2} + 2cx$$

$$-4cx = 4a^2 + 4a\sqrt{(x+c)^2+y^2}$$

$$-cx = a^2 + a\sqrt{(x+c)^2+y^2}$$

$$-cx - a^2 = a\sqrt{(x+c)^2+y^2} \quad \Rightarrow$$

$$\left(-cx - a^2\right)^2 = \left(a\sqrt{(x+c)^2+y^2}\right)^2$$

$$c^2x^2 + 2a^2cx + a^4 = a^2\left((x+c)^2 + y^2\right)$$

$$c^2x^2 + 2a^2cx + a^4 = a^2\left(x^2 + 2cx + c^2 + y^2\right)$$

$$c^2x^2 + 2a^2cx + a^4 = a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2$$

$$c^2x^2 + a^4 = a^2x^2 + a^2c^2 + a^2y^2$$

$$c^2x^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$$

$$x^2\left(c^2 - a^2\right) - a^2y^2 = a^2c^2 - a^4$$

$$x^2\left(c^2 - a^2\right) - a^2y^2 = a^2\left(c^2 - a^2\right)$$

Let $b^2 = c^2 - a^2$, and therefore $c^2 = a^2 + b^2$.

$$x^2b^2 - a^2y^2 = a^2b^2$$

Divide both sides by a^2b^2.

$$\frac{x^2b^2}{a^2b^2} - \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

2. **Equations of the hyperbolas in Part G.**

For a horizontal hyperbola centered at the origin, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Set $a = 1$ so that the vertices are at $(1, 0)$ and $(-1, 0)$. We then have: $x^2 - \frac{y^2}{b^2} = 1$.

For a desired value of e, what should b^2 be? We need to express b^2 in terms of e.

$e = \frac{c}{a}$. Since we set $a = 1$, we have: $e = \frac{c}{1} = c$.

Since $c^2 = a^2 + b^2$ and $a = 1$ and $e = c$, we have $e^2 = 1 + b^2$.

Solving for b^2, we obtain: $b^2 = e^2 - 1$.

The equation $x^2 - \frac{y^2}{b^2} = 1$ can then be written as:

$$x^2 - \frac{y^2}{e^2 - 1} = 1$$