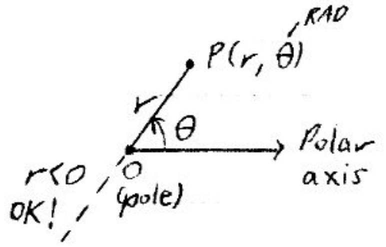
James Bern.
intro PCs in 1691
but Newton
may have used LOT
Lial 361: 1st suggested
by N (≈1671)

10.8: POLAR COORDS (PCs)

Ⓐ PCs

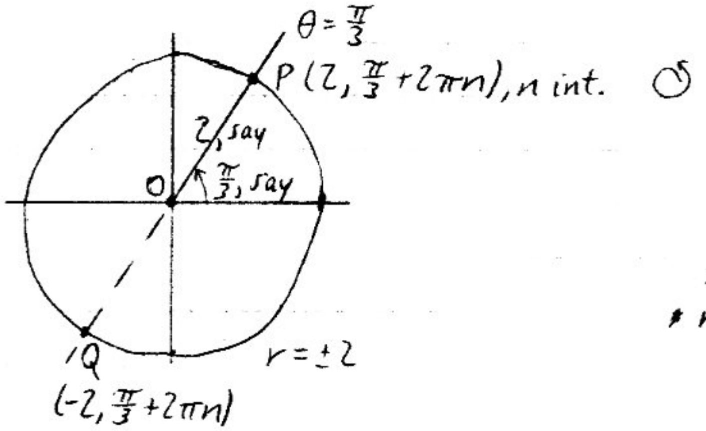
I've buried
a treasure
chest. You
can ask me
2 ?s.
Rect/Cart.
Michael Golton fix.

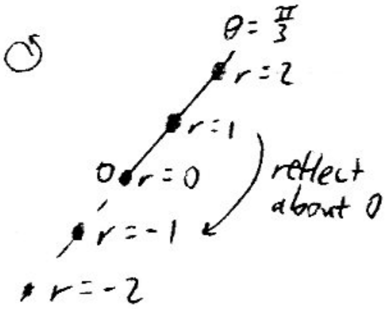


Pole O: $(0, \theta)$
any angle

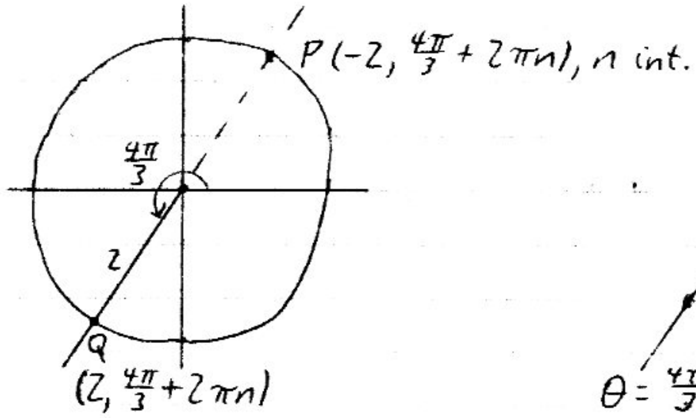
P has ∞ many PC reps.

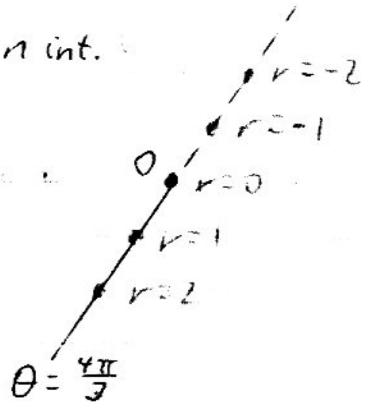
Hands!
graph $r = 2$?
what pts have
r-coord = 2?
What figure do
they form.





or





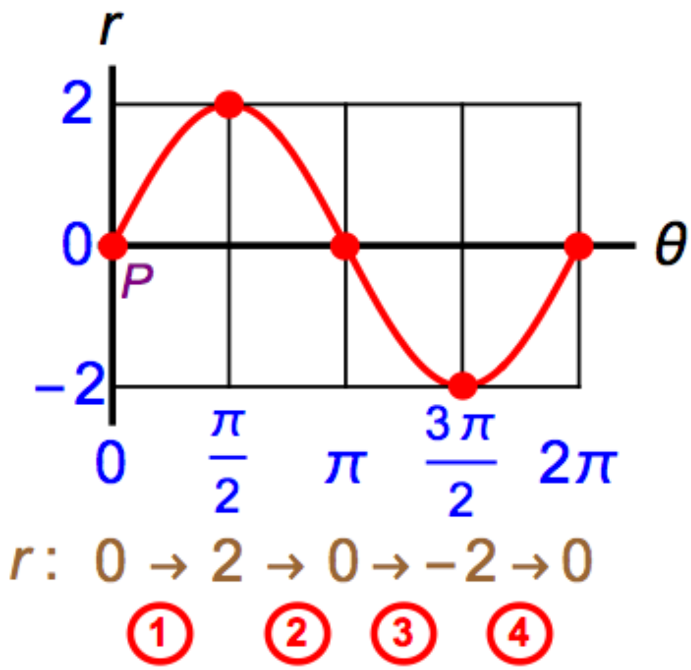
POLAR CURVES: SOME EXAMPLES

Example 1 (Graphing a Circle)

Sketch the graph of the polar equation $r = 2\sin(\theta)$, where r and θ are polar coordinates.

§ Solution

First, graph r against θ as **Cartesian** coordinates. Graph one cycle of $r = 2\sin(\theta)$.



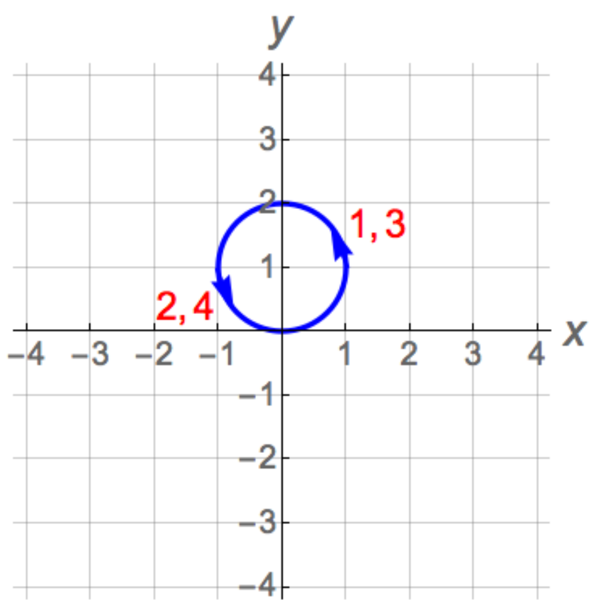
Now, graph r and θ as **polar** coordinates. Draw in Quadrant …

As $\theta: 0 \to \frac{\pi}{2}$, $r: 0 \to 2$.	(Phase 1)	I
As $\theta: \frac{\pi}{2} \to \pi$, $r: 2 \to 0$.	(Phase 2)	II
As $\theta: \pi \to \frac{3\pi}{2}$, $r: 0 \to -2$.	(Phase 3)	I (not III, because $r \le 0$)
As $\theta: \frac{3\pi}{2} \to 2\pi$, $r: -2 \to 0$.	(Phase 4)	II (not IV, because $r \le 0$)

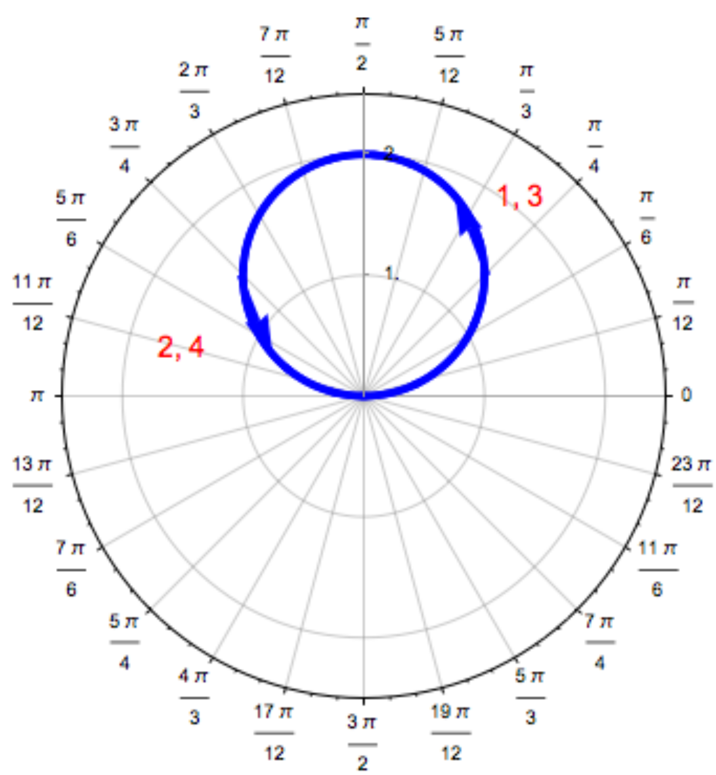
A "**new phase**" should be started whenever:

- θ changes Quadrant.
- r changes direction (from increasing to decreasing, or vice-versa).
- r changes sign.

Here is the polar curve using **Cartesian** graph paper:



Here is the polar curve using **polar** graph paper:



Obtaining the equation of the circle in Cartesian coordinates:

$$r = 2\sin(\theta) \Rightarrow \text{(Multiply both sides by } r.\text{)}$$
$$r^2 = 2r\sin(\theta)$$

We need not exclude the case $r = 0$, since 0 is in the range of the $2\sin(\theta)$ function. The **pole (origin)**, which corresponds to $r = 0$, lies on the graph.

$$x^2 + y^2 = 2y$$
$$x^2 + y^2 - 2y = 0$$
$$x^2 + (y^2 - 2y + 1) = 0 + 1 \quad \text{(CTS and balance.)}$$
$$x^2 + (y-1)^2 = 1$$

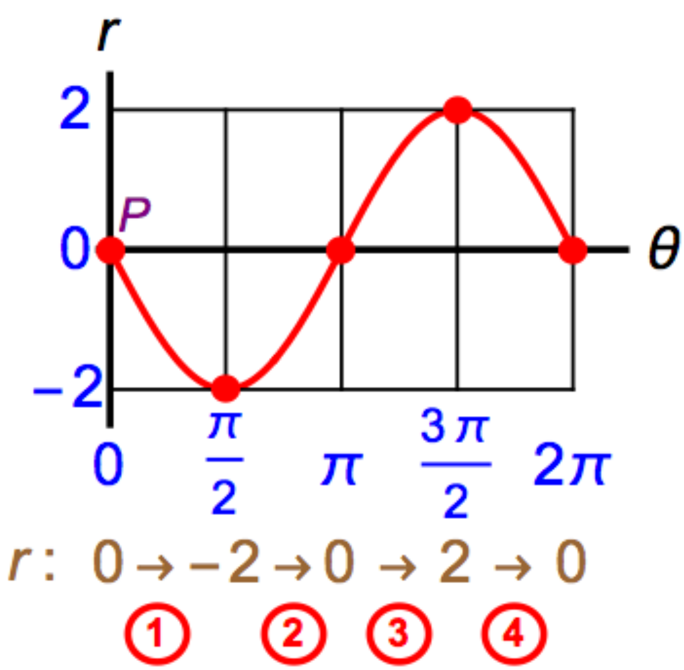
We have a **circle** of radius 1 centered at $(0, 1)$. However, this analysis does **not** indicate **orientation**. *§*

Example 2 (Graphing a Circle)

Sketch the graph of the polar equation $r = -2\sin(\theta)$, where r and θ are polar coordinates.

§ Solution

First, graph r against θ as **Cartesian** coordinates. Graph one cycle of $r = -2\sin(\theta)$.



Now, graph r and θ as **polar** coordinates. Draw in Quadrant …

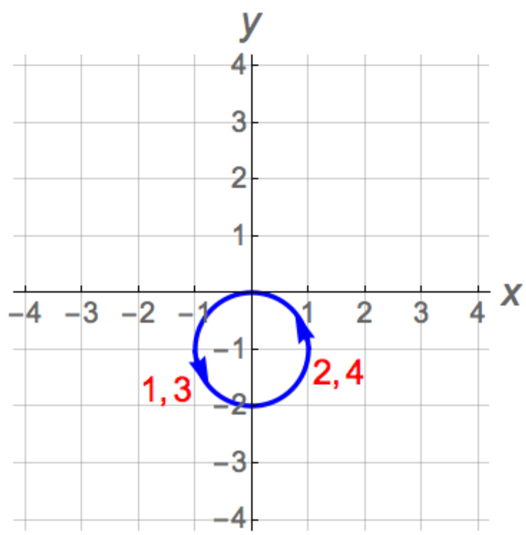
As $\theta:\ 0 \to \dfrac{\pi}{2},\quad r:\ 0 \to -2.$ (Phase 1) III (not I, because $r \le 0$)

As $\theta:\ \dfrac{\pi}{2} \to \pi,\quad r:\ -2 \to 0.$ (Phase 2) IV (not II, because $r \le 0$)

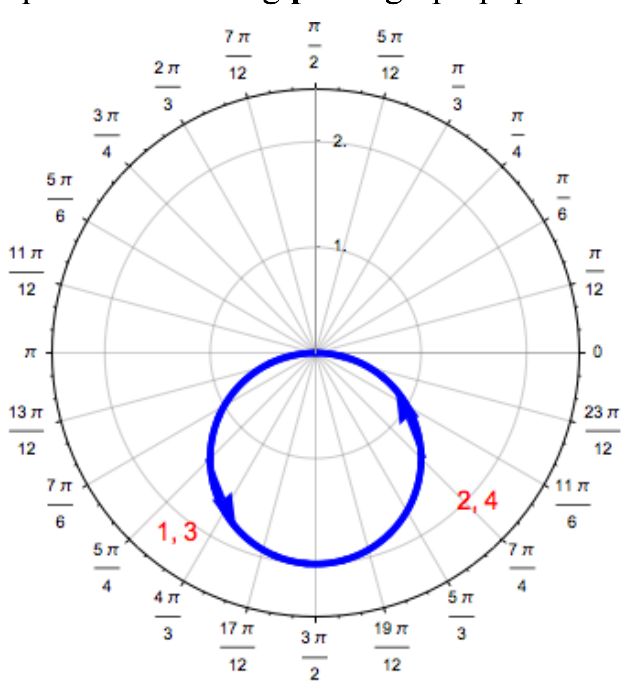
As $\theta:\ \pi \to \dfrac{3\pi}{2},\quad r:\ 0 \to 2.$ (Phase 3) III

As $\theta:\ \dfrac{3\pi}{2} \to 2\pi,\quad r:\ 2 \to 0.$ (Phase 4) IV

Here is the polar curve using **Cartesian** graph paper:



Here is the polar curve using **polar** graph paper:



§

(B) Graph of a Polar Eq.

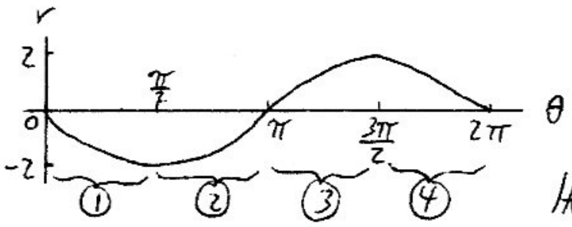
consists of all pts. (r, θ) satisfy eq.

Usual form: r or $r^2 = f(\theta)$

Ex $\quad r = -2\sin\theta$

Stewart 673 "in Cartesian coords"

Graph r vs. θ as Cartesian/rectangular coords.



$r: \; 0 \to -2 \to 0 \to 2 \to 0$

Here, "sectors" corresp. to quadrants, but be careful!

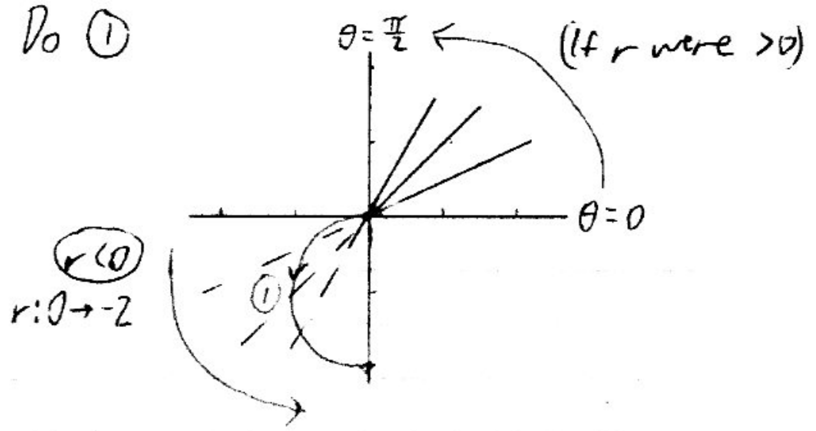
Be careful about calling these Q ($r<0$, later: ch θ))

or Table:

θ	r
0	$-2\sin(0) = 0$
$\vdots$	

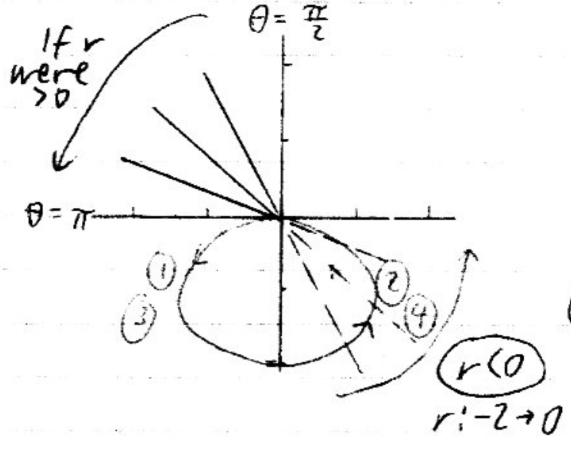
If $r>0$, we'd be going thru QI

Do ①



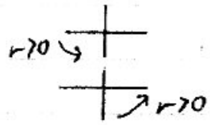
Trust me: it's a half-circle
Recognize basic forms

Do (2)



Circle — We'll show how you can prove this later.

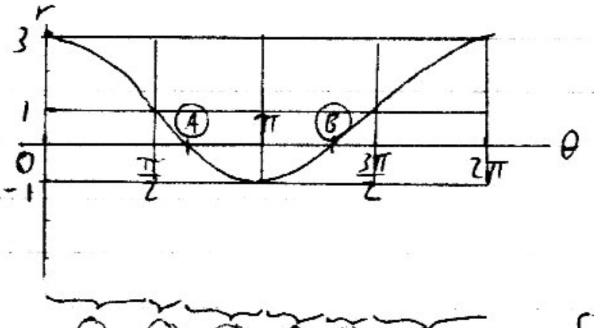
(3) retraces (1)
(4) retraces (2)



Ex

$$r = 1 + 2\cos\theta$$

Note:
$-2 \le 2\cos\theta \le 2$
$-1 \le 1 + 2\cos\theta \le 3$



(1) (2) (3) (4) (5) (6)

$r: 3 \longrightarrow 1 \to 0 \to -1 \to 0 \to 1 \longrightarrow 3$

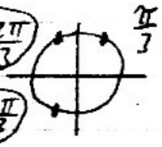
Sectors divided by:
- quadrants
- ↗ vs. ↘
- + vs. −

Find A, B

$$0 = 1 + 2\cos\theta$$
$$\cos\theta = -\frac{1}{2}$$

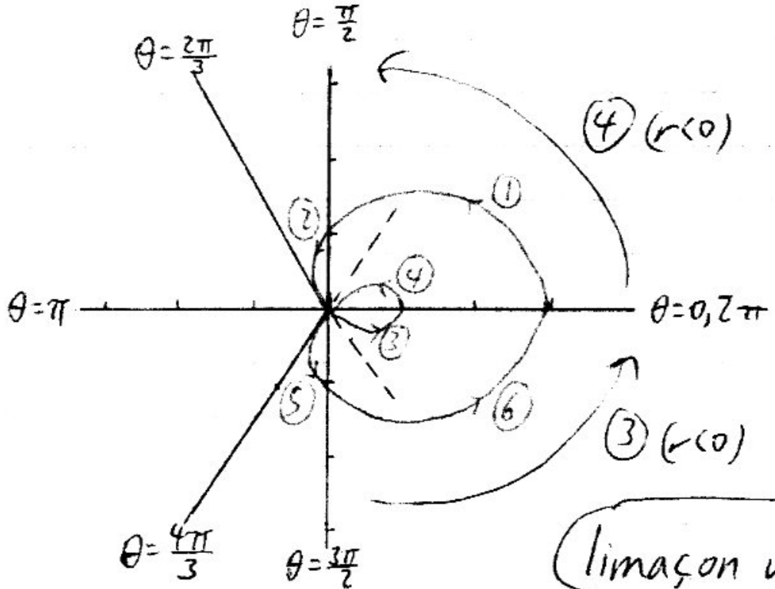
Discuss'

$A = \frac{2\pi}{3}$



$\frac{\pi}{3}: \cos = \frac{1}{2}$

$B = \frac{4\pi}{3}$



limaçon w/a loop
snail

Lee-muh-SOH"
Webster: locus set θ
fr: snail
escargot?
'prepared snail

Stewart 677
$r = 1 + c\sin\theta$
Me
$r = 1 + a\cos\theta$
$a = 1$ cardioid
$a \neq 0, 1$: limaçon
$0 < a < 1$: no loop
$\frac{1}{2} < a < 1$: dimple

$0 < a \leq \frac{1}{2}$
no dimple
oval (a.wm 677: "convex")
$a \to 0$, → circle
$r = 1 + \cos\theta$

$r = 1 + \cos\theta$

cardioid ≈ "heart"

The evolution of $r = 1 + a\cos\theta$

$r = 1 + 0.7\cos\theta$

dimpled limaçon

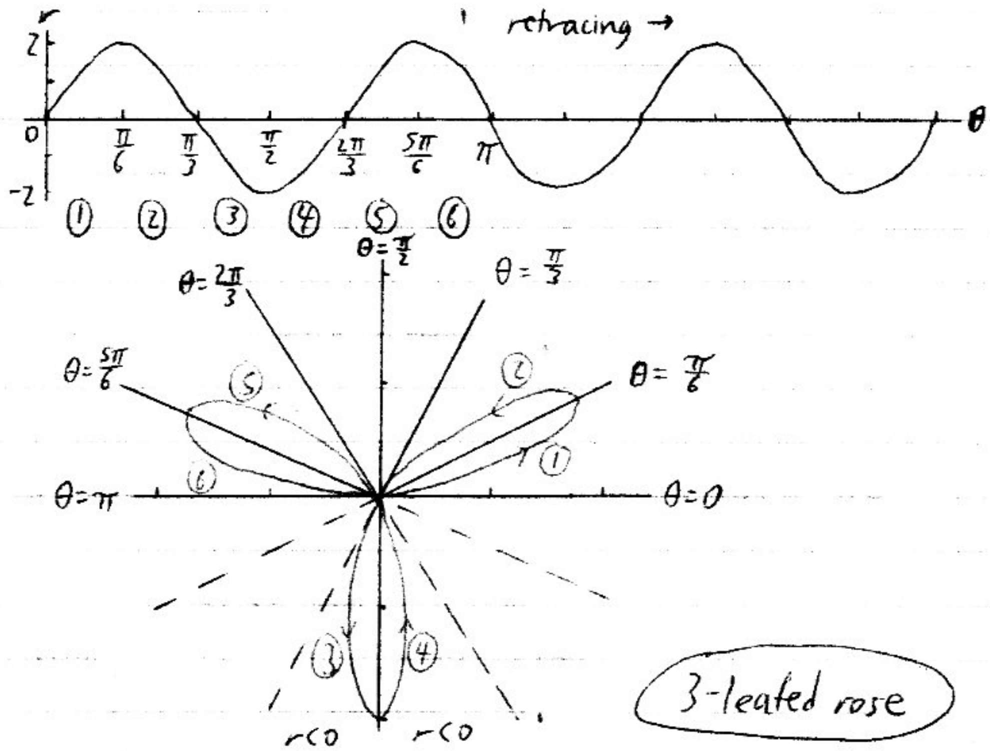
$r = 1 + 0.5\cos\theta$ (loses dimple)

loses dimple
still limaçons

$r = 1$

Circle!

Ex $r = 2\sin(3\theta)$

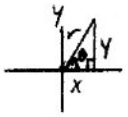


$r = a \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (n\theta)$

$a \neq 0$ if $n = 3, 5, 7, \ldots \Rightarrow n$ leaves

if $n = 2, 4, 6, \ldots \Rightarrow 2n$ leaves

© Polar Eq. ⇔ Rect. Eq.



$$\boxed{\begin{array}{l} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x},\ x \neq 0 \end{array}}$$ ⊤Watch quadrant?

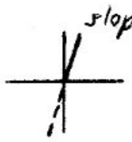
$\cos\theta = \frac{x}{r} \Rightarrow$ $\boxed{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ \text{even if } r<0 \end{array}}$
$\sin\theta = \frac{y}{r} \Rightarrow$

Ex Find a polar eq. w/same graph as $y = 6x$

$\frac{y}{x} = 6$ Also, (0,0)

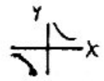
$\tan\theta = 6$

$\boxed{\theta = \tan^{-1} 6}$



Ex Find a rect. eq. w/same graph as $r^2 \sin(2\theta) = 4$, and graph it.

$$r^2(2\sin\theta\cos\theta) = 4$$
$$2\underbrace{(r\sin\theta)}_{=y}\underbrace{(r\cos\theta)}_{=x} = 4$$
$$2xy = 4$$
$$xy = 2 \quad \text{or} \quad y = \frac{2}{x}$$



Ex (again) $r = -2\sin\theta$

$$r^2 = -2\underbrace{r\sin\theta}$$
$$x^2 + y^2 = -2y$$
$$x^2 + y^2 + 2y = 0$$
$$x^2 + (y^2 + 2y + 1) = 1$$
$$x^2 + (y+1)^2 = 1$$

✓when $r=0$ OK?

Circle w/center: $(0,-1)$
radius $= 1$