CONTINUOUS PROBABILITY

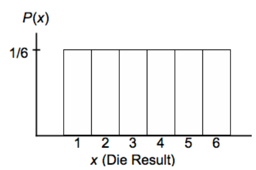
How do we work with **probabilities** when the possible outcomes have **no gaps** between them?

LESSON 19: CONTINUOUS UNIFORM DISTRIBUTIONS

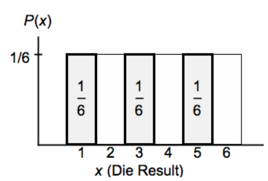
How Do We Relate Probability and Area?

PART A: PROBABILITY HISTOGRAMS and AREA

Before, we could use **probability histograms** to describe **discrete** probability distributions, such as the one for the result of a die roll:



If each bar has a **width of one unit**, then the **probability** of an event equals the **total area** of all the bars corresponding to that event. For example, the **probability** of getting an **odd** number is $\frac{3}{6}$, or $\frac{1}{2}$, the **total area of the shaded bars** in the figure below:



PART B: PROBABILITY DENSITY FUNCTIONS

When working with a **continuous** distribution, we use a probability density curve instead of a histogram. The curve is described by a probability density function.

Properties of a Probability Density Curve and its Density Function
• The **area** between a probability density curve and the x-axis is 1. • If the probability density function is f, then $f(x) \geq 0$ for all real values of x. This means that the density curve **never falls below the x-axis**.

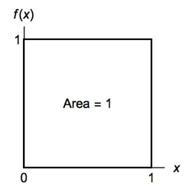
PART C: THE CONTINUOUS UNIFORM [0, 1] DISTRIBUTION

Random number generators often give values for X, where $X \sim \text{Uniform}[0,1]$. This continuous uniform distribution is perhaps the most basic continuous distribution.

The value of X can be any real number between 0 and 1.

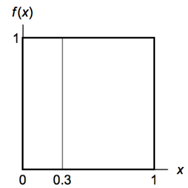
Its **probability density function** f is given by the rule $f(x)=1$, where $0 \le x \le 1$.

Its **probability density curve** is the flat top of the rectangle in the figure below:



The **area of the rectangle** is 1.

For any **continuous** distribution, the probability of any **particular value** is 0. For example, $P(X=0.3)=0$, even though we still call 0.3 a "possible" value of X. Note that the **area of the line segment** at $x=0.3$ is 0 in the figure below:



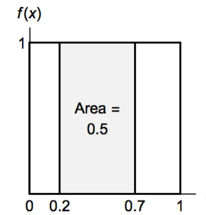
It is only possible to assign a **positive** probability to an **interval** of values for X. This probability equals the **area** under the density curve on that interval of x-values.

For a **continuous uniform distribution** on an interval, the "probability mass" is spread out evenly on that interval. Pieces of the interval that have **equal width** will correspond to **equal probabilities**.

Example 1 (Continuous Uniform [0, 1] Distribution: Probability and Area)

If $X \sim \text{Uniform}[0,1]$, then $P(0.2 < X < 0.7) = 0.5$.

This is because 0.5 is the **area** of the **shaded region** in the figure below:



Observe that the **shaded region** is a **rectangle** with base 0.5 and height 1.

§

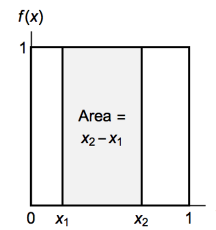
Observe that $P(X=0.2)=0$ and $P(X=0.7)=0$. For **continuous** distributions, it **does not matter** if we replace "<" with "≤" when finding a probability such as $P(0.2 < X < 0.7)$. The following statements are all true for $X \sim \text{Uniform}[0,1]$:

$$P(0.2 \le X < 0.7) = 0.5$$
$$P(0.2 < X \le 0.7) = 0.5$$
$$P(0.2 \le X \le 0.7) = 0.5$$

Probabilities for Continuous Uniform [0, 1] Distributions

Assume $X \sim \text{Uniform}[0,1]$.

If $0 \le x_1 \le x_2 \le 1$, then $P(x_1 < X < x_2) = x_2 - x_1$.



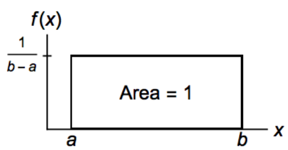
PART D: CONTINUOUS UNIFORM $[a, b]$ DISTRIBUTIONS

Let a and b be any pair of real numbers such that $a < b$.

If $X \sim \text{Uniform}\left[a, b\right]$, then X can be any value between a and b.

Its **probability density function** f is given by the rule $f(x) = \frac{1}{b-a}$, where $a \le x \le b$.

Its **probability density curve** is the flat top of the rectangle in the figure below:

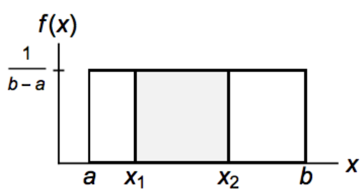


The **area of the rectangle** is 1.

Probabilities for Continuous Uniform $[a, b]$ Distributions
Assume $X \sim \text{Uniform}\left[a, b\right]$.
If $a \le x_1 \le x_2 \le b$, then $P(x_1 < X < x_2) = \frac{x_2 - x_1}{b-a}$.

This is because the **area** of the shaded rectangle below is:

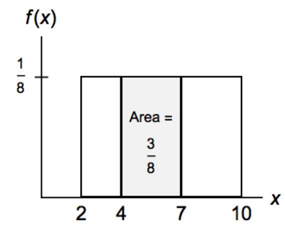
$$\text{Area} = (\text{base})(\text{height}) = (x_2 - x_1)\left(\frac{1}{b-a}\right) = \frac{x_2 - x_1}{b-a}$$



Example 2 (Continuous Uniform [a, b] Distribution: Probability and Area)

If $X \sim \text{Uniform}\left[2, 10\right]$, then

$$P(4 < X < 7) = \frac{x_2 - x_1}{b-a} = \frac{7-4}{10-2} = \frac{3}{8}, \text{ or } 0.375.$$



§

LESSON 20: THE STANDARD NORMAL DISTRIBUTION

What Do z Scores Tell Us About Normal Distributions?

PART A: THE STANDARD NORMAL DISTRIBUTION

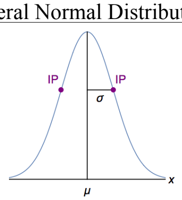
The standard normal distribution is sometimes called the *z* distribution.

It is the normal distribution with **mean 0** and **standard deviation 1**:
$Z \sim N(\mu = 0,\ \sigma = 1)$. We assume that Z is a **standard normal random variable**.

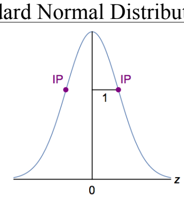
Properties of Normal Distributions

- They are **continuous**.
- A normal random variable can take on **any real number** as a value.
- The density curve is **bell-shaped**.
- The density curve is **symmetric about the mean**.
- The **standard deviation** is the distance between the "mean-line" and either inflection point ("IP"); "IP"s are points where the density curve changes from concave up (curving upward) to concave down (curving downward), or vice-versa.
- The **tails** of the curve fall off rapidly (in terms of standard deviations).

General Normal Distribution



Standard Normal Distribution



(Students usually don't have to bother with density function rules such as these:)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

(Vertical axes are often omitted when graphing density curves.)

PART B: WORKING WITH PROBABILITIES

Websites with Z tables and calculators include:
http://www.z-table.com
https://stattrek.com/online-calculator/normal.aspx

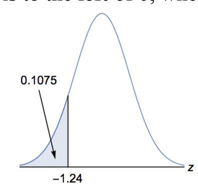
Example 1 (Z Distribution: Working with Probabilities)

Software tells you that $P(Z < -1.24) \approx 0.1075$.

- a) Sketch a figure clearly showing this fact. Shade in the relevant region.
- b) Find $P(Z > -1.24)$. Explain why and sketch a figure clearly showing this. Shade in the relevant region.
- c) Find $P(Z > 1.24)$. Explain why and sketch a figure clearly showing this. Shade in the relevant region.

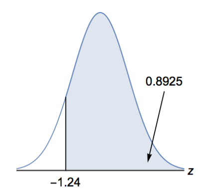
§ Solution

- a) Make sure that -1.24 is to the **left** of 0, where the curve peaks.



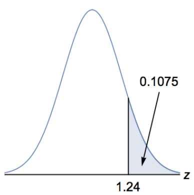
- b) We want the **complementary** probability; remember that the total area under the density curve is 1:

$$\begin{aligned} P(Z > -1.24) &= 1 - P(Z < -1.24) \\ &\approx 1 - 0.1075 \\ &\approx 0.8925 \end{aligned}$$



• c) By **symmetry about the mean**, 0:

$$P(Z > 1.24) = P(Z < -1.24)$$
$$\approx 0.1075$$



§

Example 2 (Z Distribution: Working with Probabilities)

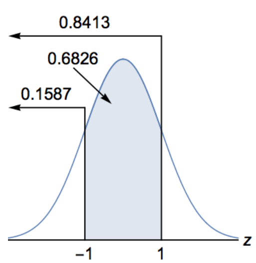
Software tells you that $P(Z < -1) \approx 0.1587$ and $P(Z < 1) \approx 0.8413$.

Find $P(-1 < Z < 1)$. Sketch a figure clearly showing this. Shade in the relevant region.

§ Solution

Make sure that -1 is to the **left** of 0 … and that 1 is to the **right** of 0.

$$P(-1 < Z < 1) \approx 0.8413 - 0.1587$$
$$\approx 0.6826$$



Note that this is consistent with our "68-95-99.7%" Empirical Rule from Lesson 9.

§

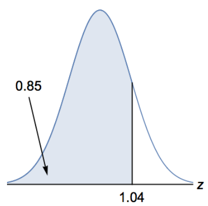
PART C: WORKING WITH PERCENTILES

Example 3 (Z Distribution: Working with Percentiles)

Software tells you that the 85th percentile of the *Z* distribution is about 1.04. Sketch a figure clearly showing this. Shade in the relevant region. Write the corresponding probability statement.

§ Solution

1.04 is the *z* score that "beats" 85% of the Z distribution.



The desired probability statement is: $P(Z < 1.04) \approx 0.85$

§