CONTINUOUS PROBABILITY

How do we work with probabilities when the possible outcomes have no gaps between them?

LESSON 19: CONTINUOUS UNIFORM DISTRIBUTIONS
How Do We Relate Probability and Area?

PART A: PROBABILITY HISTOGRAMS and AREA

Before, we could use probability histograms to describe discrete probability distributions, such as the one for the result of a die roll:

If each bar has a width of one unit, then the probability of an event equals the total area of all the bars corresponding to that event. For example, the probability of getting an odd number is \( \frac{3}{6} \), or \( \frac{1}{2} \), the total area of the shaded bars in the figure below:

PART B: PROBABILITY DENSITY FUNCTIONS

When working with a continuous distribution, we use a probability density curve instead of a histogram. The curve is described by a probability density function.

Properties of a Probability Density Curve and its Density Function

- The area between a probability density curve and the x-axis is 1.
- If the probability density function is \( f \), then \( f(x) \geq 0 \) for all real values of \( x \). This means that the density curve never falls below the x-axis.
PART C: THE CONTINUOUS UNIFORM \([0, 1]\) DISTRIBUTION

Random number generators often give values for \(X\), where \(X \sim \text{Uniform}[0, 1]\).

This continuous uniform distribution is perhaps the most basic continuous distribution.

The value of \(X\) can be any real number between 0 and 1.

Its probability density function \(f\) is given by the rule \(f(x) = 1\), where \(0 \leq x \leq 1\).

Its probability density curve is the flat top of the rectangle in the figure below:

The area of the rectangle is 1.

For any continuous distribution, the probability of any particular value is 0.

For example, \(P(X = 0.3) = 0\), even though we still call 0.3 a “possible” value of \(X\).

Note that the area of the line segment at \(x = 0.3\) is 0 in the figure below:

It is only possible to assign a positive probability to an interval of values for \(X\).

This probability equals the area under the density curve on that interval of \(x\)-values.

For a continuous uniform distribution on an interval, the “probability mass” is spread out evenly on that interval. Pieces of the interval that have equal width will correspond to equal probabilities.

Example 1 (Continuous Uniform \([0, 1]\) Distribution: Probability and Area)

If \(X \sim \text{Uniform}[0, 1]\), then \(P(0.2 < X < 0.7) = 0.5\).

This is because 0.5 is the area of the shaded region in the figure below:

Observe that the shaded region is a rectangle with base 0.5 and height 1.

Observe that \(P(X = 0.2) = 0\) and \(P(X = 0.7) = 0\). For continuous distributions, it does not matter if we replace “<” with “≤” when finding a probability such as \(P(0.2 < X < 0.7)\). The following statements are all true for \(X \sim \text{Uniform}[0, 1]\):

\[
\begin{align*}
P(0.2 \leq X < 0.7) & = 0.5 \\
P(0.2 < X \leq 0.7) & = 0.5 \\
P(0.2 \leq X \leq 0.7) & = 0.5
\end{align*}
\]

Probabilities for Continuous Uniform \([0, 1]\) Distributions

Assume \(X \sim \text{Uniform}[0, 1]\).

If \(0 \leq x_1 \leq x_2 \leq 1\), then \(P(x_1 < X < x_2) = x_2 - x_1\).
PART D: CONTINUOUS UNIFORM \([a, b]\) DISTRIBUTIONS

Let \(a\) and \(b\) be any pair of real numbers such that \(a < b\).

If \(X \sim \text{Uniform}[a, b]\), then \(X\) can be any value between \(a\) and \(b\).

Its probability density function \(f\) is given by the rule \(f(x) = \frac{1}{b - a}\), where \(a \leq x \leq b\).

Its probability density curve is the flat top of the rectangle in the figure below:

The area of the rectangle is 1.

Probabilities for Continuous Uniform \([a, b]\) Distributions
Assume \(X \sim \text{Uniform}[a, b]\).
If \(a \leq x_1 \leq x_2 \leq b\), then \(P(x_1 < X < x_2) = \frac{x_2 - x_1}{b - a}\).

This is because the area of the shaded rectangle below is:

\[
\text{Area} = (\text{base})(\text{height}) = (x_2 - x_1) \left( \frac{1}{b - a} \right) = \frac{x_2 - x_1}{b - a}
\]

Example 2 (Continuous Uniform \([a, b]\) Distribution: Probability and Area)

If \(X \sim \text{Uniform}[2, 10]\), then

\[
P(4 < X < 7) = \frac{x_2 - x_1}{b - a} = \frac{7 - 4}{10 - 2} = \frac{3}{8}, \text{ or } 0.375.
\]
LESSON 20: THE STANDARD NORMAL DISTRIBUTION

What Do z Scores Tell Us About Normal Distributions?

PART A: THE STANDARD NORMAL DISTRIBUTION

The standard normal distribution is sometimes called the z distribution.

It is the normal distribution with mean 0 and standard deviation 1:

\[ Z \sim N(\mu = 0, \sigma = 1) \]

We assume that Z is a standard normal random variable.

Properties of Normal Distributions

- They are continuous.
- A normal random variable can take on any real number as a value.
- The density curve is bell-shaped.
- The density curve is symmetric about the mean.
- The standard deviation is the distance between the “mean-line” and either inflection point (“IP”); “IP”s are points where the density curve changes from concave up (curving upward) to concave down (curving downward), or vice-versa.
- The tails of the curve fall off rapidly (in terms of standard deviations).

General Normal Distribution

\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

(Students usually don’t have to bother with density function rules such as these.)

Standard Normal Distribution

\[ f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \]

Vertical axes are often omitted when graphing density curves.

PART B: WORKING WITH PROBABILITIES

Websites with Z tables and calculators include:

http://www.z-table.com
https://stattrek.com/online-calculator/normal.aspx

Example 1 (Z Distribution: Working with Probabilities)

Software tells you that \( P(Z < -1.24) \approx 0.1075 \).

- a) Sketch a figure clearly showing this fact. Shade in the relevant region.
- b) Find \( P(Z > -1.24) \). Explain why and sketch a figure clearly showing this. Shade in the relevant region.
- c) Find \( P(Z > 1.24) \). Explain why and sketch a figure clearly showing this. Shade in the relevant region.

Solution

- a) Make sure that -1.24 is to the left of 0, where the curve peaks.

\[ P(Z > -1.24) = 1 - P(Z < -1.24) \]

\[ = 1 - 0.1075 \]

\[ = 0.8925 \]

- b) We want the complementary probability; remember that the total area under the density curve is 1:

\[ P(Z > -1.24) = 1 - P(Z < -1.24) \]

\[ = 1 - 0.1075 \]

\[ = 0.8925 \]
• c) By symmetry about the mean, 0:
\[ P(Z > 1.24) = P(Z < -1.24) \]
\[ = 0.1075 \]

Example 2 (Z Distribution: Working with Probabilities)

Software tells you that \( P(Z < -1) \approx 0.1587 \) and \( P(Z < 1) \approx 0.8413 \).

Find \( P(-1 < Z < 1) \). Sketch a figure clearly showing this. Shade in the relevant region.

Solution

Make sure that \(-1\) is to the left of 0 … and that 1 is to the right of 0.

\[ P(-1 < Z < 1) = 0.8413 - 0.1587 = 0.6826 \]

Note that this is consistent with our “68-95-99.7%” Empirical Rule from Lesson 9.

PART C: WORKING WITH PERCENTILES

Example 3 (Z Distribution: Working with Percentiles)

Software tells you that the 85\textsuperscript{th} percentile of the \( Z \) distribution is about 1.04. Sketch a figure clearly showing this. Shade in the relevant region. Write the corresponding probability statement.

Solution

1.04 is the \( z \) score that “beats” 85% of the \( Z \) distribution.

The desired probability statement is: \( P(Z < 1.04) = 0.85 \)