CH. 9: ∫ TECHNIQUES

9.1: ∫ BY PARTS (like a product rule for ∫s)

∫ product usu. ≠ product of ∫s

Ex (#2) Evaluate $\int \underbrace{x \sin x \, dx}_{\text{Split into } u,\, dv}$

must include dx

Book's general strategy: (Hope du is nice)

Make dv as big/complex as you can?

Strategy: $\int \underbrace{\begin{pmatrix}\text{Easy} \\ \text{to } D_x\end{pmatrix}}_{=u} \underbrace{\begin{pmatrix}\text{Biggest piece} \\ \text{that's easy to } \int\end{pmatrix}}_{=dv}$

X - would you rather ∫ or D_x

$$\int \underbrace{x}_{u} \underbrace{\sin x \, dx}_{dv}$$

Let $u = x$ — $\cdot$ Let $dv = \sin x \, dx$ ∫s

$du = dx$ — $-\int\cdot$ — $v = -\cos x$

Don't need +C

$$\boxed{\int \text{by Parts}: \int u \, dv = uv - \int v \, du}$$

$$\begin{aligned}\int x \sin x \, dx &= -x \cos x - \int -\cos x \, dx \\ &= -x \cos x + \int \cos x \, dx \\ &= \boxed{-x \cos x + \sin x + \underbrace{C}_{\text{Need!}}}\end{aligned}$$

Do 1

Can D_x to ✓.

<u>What if...</u> $\int \underbrace{x \underbrace{\sin x}_{u} dx}_{dv}$

$$u = \sin x \qquad dv = x\,dx$$
$$du = \cos x\,dx \qquad v = \frac{x^2}{2}$$

$$\frac{1}{2}x^2 \sin x - \underbrace{\int \frac{1}{2}x^2 \cos x\,dx}_{\text{WORSE!}}$$

x: You'd rather D_x than $\int$
$\sin x$: OK to D_x <u>or</u> $\int$

<u>Ex</u> $\int \underbrace{x \underbrace{\ln x}_{u} dx}_{dv}$ Hard to $\int \ln x$!!
(We'll do this next.)

$$u = \ln x \qquad dv = x\,dx$$
$$du = \frac{1}{x}\,dx \qquad v = \frac{x^2}{2}$$

$$\begin{aligned}\int x \ln x\,dx &= \frac{1}{2}x^2 \ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + C \\ &= \boxed{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}\end{aligned}$$

Do 1

Ex 3, p. 458

Ex $\int \underbrace{\ln x}_{u} \underbrace{dx}_{dv}$

$u = \ln x \qquad dv = dx$
$du = \frac{1}{x}\,dx \qquad v = x$ (·, -s.)

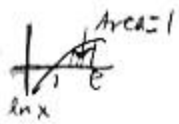
$$\int \ln x\,dx = x\ln x - \int x\left(\tfrac{1}{x}\right)dx$$
$$= x\ln x - \int dx$$
$$= \boxed{x\ln x - x + C}$$

Doesn't help us develop S by parts

If we ever do
$u = \quad dv = \ln x\,dx$
$du = \quad v = x\ln x - x$ (·, -s.)
YUK!

Ex $\int_1^e \ln x\,dx = [x\ln x - x]_1^e$
$= [e\underbrace{\ln e}_{=1} - e] - [1\underbrace{\ln 1}_{=0} - 1]$
$= \underbrace{[e-e]}_{=0} - [0-1]$
$= \boxed{1}$

What do definite Ss correspond to?



Good news and bad news...

Ex (#18) $\int \underbrace{e^{3x}}_{dv} \underbrace{\cos(2x)}_{u}\,dx$

CAN SKIP - WE'LL DO A SHORTCUT LATER!

$u = \cos(2x) \qquad dv = e^{3x}\,dx$
$du = -\sin(2x)\cdot 2\,dx \qquad v = \frac{1}{3}e^{3x}$
$= -2\sin(2x)\,dx$ (·, -s.)

$$\int e^{3x}\cos(2x)\,dx = \tfrac{1}{3}e^{3x}\cos(2x) - \int \left(\tfrac{1}{3}e^{3x}\right)(-2\sin(2x))\,dx$$

$$= \tfrac{1}{3}e^{3x}\cos(2x) + \tfrac{2}{3}\underbrace{\int e^{3x}\sin(2x)\,dx}_{\bigstar}$$

$\bigstar$ (DO PARTS AGAIN!)

$u = e^{3x}$ works, too vs Ex. 5

$$\bigstar = \int \underbrace{e^{3x}}_{u}\,\underbrace{\sin(2x)\,dx}_{dv}$$

$$u = \sin(2x) \qquad dv = e^{3x}\,dx$$
$$du = 2\cos(2x)\,dx \qquad v = \tfrac{1}{3}e^{3x}$$

$$\bigstar = \tfrac{1}{3}e^{3x}\sin(2x) - \int \left(\tfrac{1}{3}e^{3x}\right)(2\cos(2x))\,dx$$
$$= \tfrac{1}{3}e^{3x}\sin(2x) - \tfrac{2}{3}\int e^{3x}\cos(2x)\,dx$$

≈ semi-circular, self-referential

$$\underbrace{\int e^{3x}\cos(2x)\,dx}_{I} = \tfrac{1}{3}e^{3x}\cos(2x) + \tfrac{2}{3}\left[\tfrac{1}{3}e^{3x}\sin(2x) - \tfrac{2}{3}\underbrace{\int e^{3x}\cos(2x)\,dx}_{I}\right]$$

$$I = \tfrac{1}{3}e^{3x}\cos(2x) + \tfrac{2}{9}e^{3x}\sin(2x) - \tfrac{4}{9}I$$

Solve for I

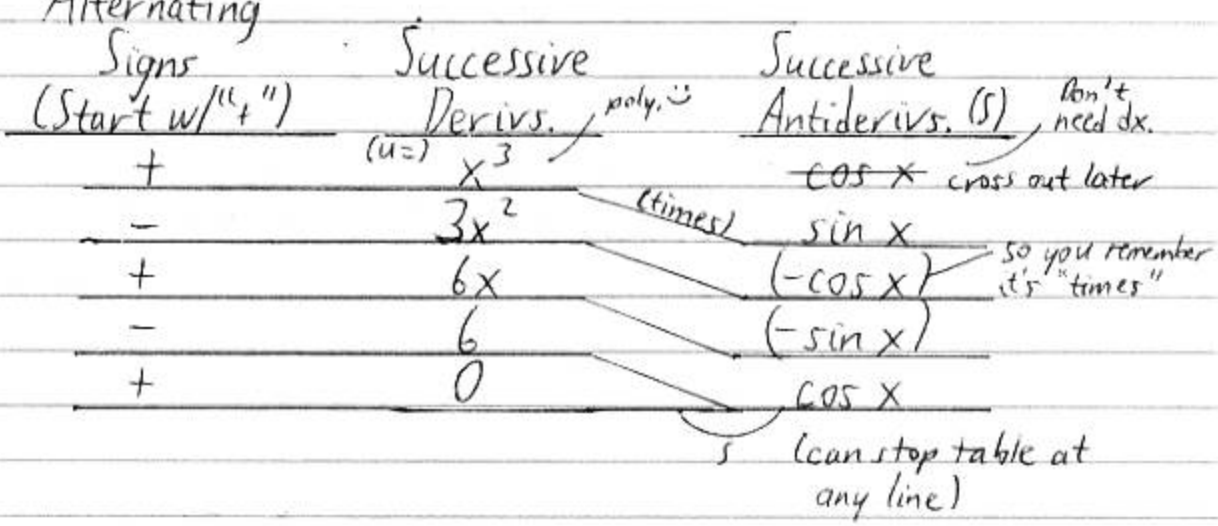
$$\tfrac{13}{9}I = \quad , \qquad +C$$
$$I = \tfrac{9}{13}\left[\tfrac{1}{3}e^{3x}\cos(2x) + \tfrac{2}{9}e^{3x}\sin(2x)\right] + C$$

$$\boxed{I = \tfrac{3}{13}e^{3x}\cos(2x) + \tfrac{2}{13}e^{3x}\sin(2x) + C}$$

Need!

SHORTCUT: TABULAR ∫ by PARTS ("TURBO METHOD")

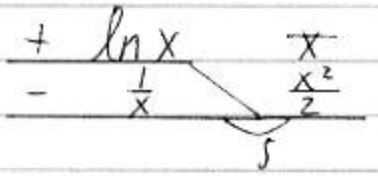
Ex $\int x^3 \cos x \, dx$ (Need to use ∫ by Parts more than once!)



$$x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + \underbrace{\int 0 \cos x \, dx}_{\int \text{ of last line}}$$
$$= 0 + C$$

$$\boxed{x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C}$$

Old Ex $\int x \ln x \, dx$



$$\frac{1}{2} x^2 \ln x - \int \frac{1}{x} \cdot \frac{x^2}{2} \, dx$$
$$= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x \, dx$$
etc.

Old Ex $\int e^{3x}\cos(2x)\,dx$

$u = e^{3x}$ works, too vs. Ex 5

$+$	$\cos(2x)$	e^{3x}
$-$	$(-2\sin(2x))$	$\frac{1}{3}e^{3x}$
$+$	$(-4\cos(2x))$	$\frac{1}{9}e^{3x}$
		$\int$ (Remainder $\int$)

Let's stop table here. (We'll see why.)

≈ semi-circular, self-referential

$$\underbrace{\int e^{3x}\cos(2x)\,dx}_{\text{Call "I"}} = \frac{1}{3}e^{3x}\cos(2x) + \frac{2}{9}e^{3x}\sin(2x) - \frac{4}{9}\underbrace{\int e^{3x}\cos(2x)\,dx}_{I}$$

$$I = \frac{1}{3}e^{3x}\cos(2x) + \frac{2}{9}e^{3x}\sin(2x) - \frac{4}{9}I$$

Solve for I.
Add $\frac{4}{9}I$ to both sides.

$$\frac{13}{9}I = (\qquad) + C$$

$\frac{9}{13}C \Rightarrow$ just call "C"

$$I = \frac{9}{13}\left[\frac{1}{3}e^{3x}\cos(2x) + \frac{2}{9}e^{3x}\sin(2x)\right] + C$$

$$\boxed{I = \frac{3}{13}e^{3x}\cos(2x) + \frac{2}{13}e^{3x}\sin(2x) + C}$$

Compare w/ the long way, which I worked out! (9.1.3–9.1.4)

9.2: TRIG $\int$s

Ⓐ Review IDs

Pythagorean

$$\sin^2 x + \cos^2 x = 1$$

$\div \sin^2 x$: $1 + \cot^2 x = \csc^2 x$

$\div \cos^2 x$: $\tan^2 x + 1 = \sec^2 x$

$(\cup\ \cap)^2$

Power-Reducing IDs (PRIs)

sin is bad

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

OK: neg or frac

Ⓑ $\int \sin^m x \cos^n x \, dx$

Larson +
We like the odd power-hurt ones we love
Beat him up
low, + int. exp.

Case 1 (m and/or n is odd, >0)

Rip apart the [easier] odd power.
if both odd

Rip out

- Peel one sin x or cos x factor.
- Rewrite the rest using a Pyth. ID.

Let $u = \sin x$ or $\cos x$
Who was *not* peeled? "Other guy"

Then, $du = \pm$ (peeled) dx.

Ex $\int \sin^3 x \, dx \quad (n=0)$

$= \int \underbrace{\sin^2 x}_{\text{Use Pyth. Id.}} \cdot \overbrace{\sin x}^{\text{Peel}} \, dx$

$= \int (1 - \cos^2 x) \sin x \, dx$

OK:
$u = -\cos x$
$du = \sin x \, dx$
but $u \in (-\cos x)$ at end

Other guy "Embedded"
Let $u = \cos x$
$du = -\sin x \, dx \Rightarrow \sin x \, dx = -du$

($= -\int (1 - \cos^2 x)(\underbrace{-\sin x \, dx}_{du})$) can skip

$= -\int (1 - u^2) \, du$ Note ($= \int (u^2 - 1) \, du$ "Switch Rule")

$= -\left[u - \frac{u^3}{3}\right] + C$

Am I done?

$= -u + \frac{1}{3}u^3 + C$

Sub back

Mathem gives $\frac{1}{12}(-9\cos x + \cos(3x))$ - another form

$= \boxed{-\cos x + \frac{1}{3}\cos^3 x + C}$

Ex (#4) $\int \cos^7 x \, dx$

How is $\cos^6 x$ related to $\cos^2 x$?

$$= \int \cos^6 x \cdot \overbrace{\cos x}^{\text{Peel}} \, dx$$

$$= \int (\underbrace{\cos^2 x}_{\text{Use Pyth. ID.}})^3 \cos x \, dx$$

$$= \int (1-\sin^2 x)^3 \cos x \, dx$$

Let $u = \overbrace{\sin x}^{\text{Other guy}}$

$du = \cos x \, dx$

We care about exp!

If had $(1-u)^{10}$, Can use $w = 1-u$, but not naturally from Pyth. IDs

$$= \int (1-u^2)^3 \, du$$

Binomial Thm.:

$$(\underset{(1)}{a}+\underset{(-u^2)}{b})^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Pascal's Δ

Row	
0	1
1	1 1
2	1 2 1
3	1 3 3 1

coeffs.

$$= \int [(1)^3 + 3(1)^2(-u^2) + 3(1)(-u^2)^2 + (-u^2)^3] \, du$$

$$= \int [1 - \underbrace{3u^2}_{\text{or Guess-and-}\checkmark} + 3u^4 - u^6] \, du$$

$$= u - \cancel{3}\left(\frac{u^3}{\cancel{3}}\right) + 3\left(\frac{u^5}{5}\right) - \frac{u^7}{7} + C$$

Sub back

$$= \boxed{\sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C}$$

80 is numerically intimidating, but it doesn't hurt us. 1,000,000 OK

Ex $\int \sin^5 x \cos^{80} x \, dx$ (odd)

$= \int \sin^4 x \cos^{80} x \cdot \sin x \, dx$ (peel)

$= \int (\sin^2 x)^2 \cos^{80} x \cdot \sin x \, dx$

Use Pyth. ID.

$= \int (1-\cos^2 x)^2 \cos^{80} x \cdot \sin x \, dx$

Let $u = \cos x$

$du = -\sin x \, dx$

$\Rightarrow \quad \sin x \, dx = -du$

$= -\int (1-u^2)^2 u^{80} \, du$

$= -\int (1-2u^2+u^4) u^{80} \, du$

$= -\int (u^{80} - 2u^{82} + u^{84}) \, du$

$= -\left[\frac{u^{81}}{81} - 2\left(\frac{u^{83}}{83}\right) + \frac{u^{85}}{85}\right] + C$

$= -\frac{1}{81}u^{81} + \frac{2}{83}u^{83} - \frac{1}{85}u^{85} + C$

Sub back.

Dangerous to flip signs at same time. I guess I can't walk + chew gum at same time.

Mathem → ugly constants!

$= \boxed{-\frac{1}{81}\cos^{81} x + \frac{2}{83}\cos^{83} x - \frac{1}{85}\cos^{85} x + C}$

Case 2 (m, n both even, ≥ 0)

Power-Reducing IDs

Use PRIs.

Ex $\int \cos^4 x \, dx$

$= \int (\cos^2 x)^2 \, dx$

(Use PRI.)

$= \int \left[\frac{1+\cos(2x)}{2}\right]^2 dx$

cos is the θ_x of what?

$= \frac{1}{4} \int [1 + 2\cos(2x) + \cos^2(2x)] \, dx$

$= \frac{1}{4} \left[x + \underbrace{\sin(2x)}_{\text{Guess-and-✓ OK}} + \int \underbrace{\cos^2(2x)}_{\text{PRI: } \cos^2 u = \frac{1+\cos(2u)}{2},\ u = 2x} dx \right]$ (4x)

$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{4} \int \frac{1+\cos(4x)}{2} \, dx$

Try sin(4x) What's the θ_x

$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}\left[x + \frac{1}{4}\sin(4x)\right] + C$

$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{32}\sin(4x) + C$

$= \boxed{\frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C}$

World not just made up of sin, cos → tan, cot, Compa-Coompar

we like his power, like Joe Mil.

Ex 5, p. 466: $\int \tan^{odd} \sec^{odd}$ csc odd, though

© More $\int$s

We like:		$\int \tan^m x \sec^n x \, dx$	$\int \cot^m x \csc^n x \, dx$
m odd, +	$u =$ Peel	$\sec x$ $\sec x \tan x$	$\csc x$ $(-) \csc x \cot x$
n even, +	$u =$ Peel	$\tan x$ $\sec^2 x$	$\cot x$ $(-) \csc^2 x$

Think: "tan is odd, sec is even" →

$u =$ Other guy

$du = \underbrace{\sim}_{(-?)\text{Peel}} dx$

Use Pyth. ID to make "u"s.

Like (B) (sin, cos Exs from (B))

If m odd and n even $\int \tan^3 x \sec^6 x \, dx$

Like better 3<6 Let u = other = sec x

Why we like sec even

Ex $\int \tan^{2/3} x \sec^{6} x \, dx$ (6: even, +)

Let $u = \tan x$.

$du = \sec^2 x \, dx$

$$= \int \tan^{2/3} x \underbrace{\sec^4 x}_{} \cdot \overbrace{\sec^2 x}^{\text{Peel}} \, dx$$

Use Pyth. ID to make "u"s ("tan x"s).

Use: $\tan^2 x + 1 = \sec^2 x$.

$$= \int \tan^{2/3} x \, (\sec^2 x)^2 \cdot \sec^2 x \, dx$$

$$= \int \tan^{2/3} x \, (\tan^2 x + 1)^2 \cdot \sec^2 x \, dx$$

$u = \tan x$
$du = \sec^2 x \, dx$

$= \int u^{2/3} (u^2+1)^2 \, du$

$= \int u^{2/3} (u^4 + 2u^2 + 1) \, du$

$= \int (u^{14/3} + 2u^{8/3} + u^{2/3}) \, du$

$= \frac{u^{17/3}}{17/3} + 2\left(\frac{u^{11/3}}{11/3}\right) + \frac{u^{5/3}}{5/3} + C$

$= \boxed{\frac{3}{17}\tan^{17/3} x + \frac{6}{11}\tan^{11/3} x + \frac{3}{5}\tan^{5/3} x + C}$

Reminder:

		$\int \tan^m x \sec^n x \, dx$	$\int \cot^m x \csc^n x \, dx$
m	$u =$	$\sec x$	$\csc x$
odd, +	Peel	$\sec x \tan x$	$(-) \csc x \cot x$
n	$u =$	$\tan x$	$\cot x$
even, +	Peel	$\sec^2 x$	$(-) \csc^2 x$

Ex $\int \cot^{\overset{\text{odd,+}}{3}} x \csc^7 x \, dx$

Let $u = \csc x$.
$du = -\csc x \cot x \, dx$.

$= \int \underbrace{\cot^2 x}\, \csc^6 x \cdot \overbrace{\csc x \cot x}^{\text{peel}} \, dx$

Use Pyth. ID to make "u"s ("csc x"s)
Use: $1 + \cot^2 x = \csc^2 x$
$\Rightarrow \quad \cot^2 x = \csc^2 x - 1$

Book throws around

$$= \int (\csc^2 x - 1)\csc^6 x \cdot \csc x \cot x \, dx$$

$$u = \csc x$$
$$du = -\csc x \cot x \, dx$$

or $\int (1-u^2)u^6 \, du$

$$= -\int (u^2 - 1)u^6 \, du$$

$$= -\int (u^8 - u^6) \, du$$

$$= -\left[\frac{u^9}{9} - \frac{u^7}{7}\right] + C$$

$$= -\frac{1}{9}u^9 + \frac{1}{7}u^7 + C$$

$$= \boxed{-\frac{1}{9}\csc^9 x + \frac{1}{7}\csc^7 x + C}$$

PRIs not nice: $\tan^2 x = \frac{1-\cos(2x)}{1+\cos(2x)}$, $\csc^2 x$: reciprocal of $\sin^2 x$ ID (right side)

$\frac{2}{1-\cos(2x)}$

cot

$\int \tan^{(m)} x \, dx$ (even, +) — Peel $\tan^2 x$. Pyth. ID ↗

csc

$\int \sec^{(n)} x \, dx$ (odd, +) — Parts!

Parties

If csc cas

If simp expr w/ tan, cot, csc

Trick: → sin, cos (if have mix of csc, cot, ...)

9.3: TRIG SUBS

(A) Intro

8.2: Proof $D_x(\sin^{-1}x)$ using Implicit.

Integrable, furthermore

Ex Show $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$

Domain $= (-1, 1)$ are "sin values"

θ in $(-\frac{\pi}{2}, \frac{\pi}{2})$

Range of $\sin^{-1} = (-\frac{\pi}{2}, \frac{\pi}{2})$

Let $x = \sin\theta$, θ in

Then, $\theta = ?$

$\Rightarrow \theta = \sin^{-1}x$

Sub: $x = \sin\theta$
$dx = \cos\theta\,d\theta$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$$

What's $\sqrt{x^2}$?

$\sqrt{\cos^2\theta}$ by Pyth. ID
$= |\cos\theta|$ (bec. $\sqrt{A^2} = |A|$)
$= \cos\theta$ ~~$\cos\theta > 0$~~

$$= \int \frac{1}{\cos\theta}\cos\theta\,d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

Sub back!

$$= \boxed{\sin^{-1}x + C}$$

Assume "a" real, +.

If the integrand has	Sub
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$
or $a^2 + x^2$	
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$

Use a Pyth. Id.

Edwards has details, but uses diff ranges.

Area of ellipse Stewart 491

Set up and do a date or $x = a\cosh\theta$ Stewart 493

Note

Really, sub rule requires $\theta = \sin^{-1}\left(\frac{x}{a}\right)$, $|x| \le a$ relevant to $\int$.

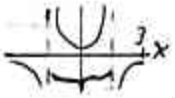
(Arc length!) θ in range of $\sin^{-1} = [-\frac{\pi}{2}, \frac{\pi}{2}]$

Swokowski defines the range of $\csc^{-1}$ to be $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ (p. 453, #32), so $\sin^{-1}\left(\frac{x}{a}\right) = \csc^{-1}\left(\frac{a}{x}\right)$ if $|x| \le a$ and $x \neq 0$.

However, $\sec^{-1}\left(\frac{x}{a}\right) \neq \cos^{-1}\left(\frac{a}{x}\right)$ for $x < -a$, because of the very different ranges of $\sec^{-1}$, $\cos^{-1}$. (Also for $\tan^{-1}$, $\cot^{-1}$.)

Ex $\int \frac{1}{1 - x^2}\,dx$

Domain = {real $x \mid x \neq \pm 1$}



~~$x = \sin\theta$~~ 3 is not a sin value!

(9.4)

Ex $\int \frac{1}{1 + x^2}\,dx$

Domain = $\mathbb{R}$

$x = \tan\theta$ ✓

Every real x is a "tan value."

Is 3 a tan value?

$= \tan^{-1} x + C$

Ex $\int \frac{x}{1 + x^2}\,dx$

Easier: $u = 1 + x^2$

$du = 2x\,dx$

In 9.3, we can ignore domain, range, 11 issues - they work out!

(B) Exs

Helps w/ II

Ex (#6) $\int \frac{1}{x^3(\sqrt{x^2-25})}\,dx$ ~~$u = x^2-25$?~~

$7 = (\sqrt{7})^2$

Integrand has $\sqrt{(x)^2-(5)^2}$

Sub: $x = 5\sec\theta$
$dx = 5\sec\theta\tan\theta\,d\theta$

$dx \to d\theta$?? NO!

$$= \int \frac{1}{(5\sec\theta)^3(\sqrt{(5\sec\theta)^2-25})}(5\sec\theta\tan\theta\,d\theta)$$

Can skip:
$= \sqrt{25\sec^2\theta-25}$
$= \sqrt{25(\sec^2\theta-1)}$
$= 5\sqrt{\tan^2\theta}$ (by Pyth. ID)
$= 5\tan\theta$ (ignore ||)

$d\theta$ OK: $\int \frac{dy}{u}$ OK (Just like)

$$= \int \frac{5\sec\theta\tan\theta\,d\theta}{125\sec^3\theta\cdot 5\tan\theta}$$

$$= \frac{1}{125}\int \frac{1}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{125}\int \underbrace{\cos^2\theta}_{\text{Use a PRI.}}\,d\theta$$

$$= \frac{1}{125}\int \frac{1+\cos(2\theta)}{2}\,d\theta$$

$$\left(= \frac{1}{250}\int (1+\cos(2\theta))\,d\theta\right)$$

$$= \frac{1}{250}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C$$

(Use a Double-Angle ID)

$= 2\sin\theta\cos\theta$

$$= \frac{1}{250}\left[\theta + \sin\theta\cos\theta\right] + C$$

Are we done? No way!

Go back to x!!

$\underline{\theta = ?}$

$x = 5\sec\theta$

$\frac{x}{5} = \sec\theta$

$\theta = \sec^{-1}\left(\frac{x}{5}\right)$ (Ignore domain, range issues.)

We want alg. expres. wherever possible.

$\underline{\cos\theta = ?}$ $\cos\left(\sec^{-1}\left(\frac{x}{5}\right)\right)$ not good enough!

$\sec\theta = \frac{x}{5}$

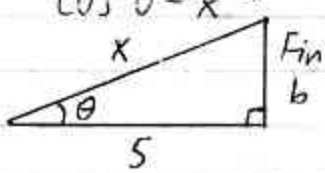
$\cos\theta = \frac{5}{x}$

$\underline{\sin\theta = ?}$

Method 1 (Set up a right triangle.)

$\cos\theta = \frac{5}{x} \Rightarrow$

Use b: $x = a\sec\theta$



(You can assume θ is acute.)

$5^2 + b^2 = x^2$

$b^2 = x^2 - 25$

Looks familiar? Always works! Sometimes w/a fun sub, you intro ✓

$b = \pm\sqrt{x^2 - 25}$ (In 1st ∫)

Is this x-5?

$\sin\theta = \frac{b}{x} = \frac{\sqrt{x^2-25}}{x}$

Method 2 (Pyth. ID)

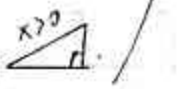
$$\sin^2\theta = 1-\cos^2\theta, \text{ Take "+" root.}$$

SKIP

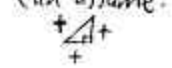
$$= 1-\left(\frac{5}{x}\right)^2$$
$$= 1-\frac{25}{x^2}$$
$$= \frac{x^2-25}{x^2}$$

Can assume: + + +

$$\sin\theta = \pm\sqrt{\frac{x^2-25}{x^2}}$$
$$= \frac{\sqrt{x^2-25}}{x}$$



$x>0$



$$\frac{1}{250}\left[\theta + \sin\theta\cos\theta\right] + C$$

$$= \frac{1}{250}\left[\sec^{-1}\left(\frac{x}{5}\right) + \frac{\sqrt{x^2-25}}{x}\cdot\frac{5}{x}\right] + C$$

Don't you miss u-subs now?

$$= \boxed{\frac{1}{250}\left[\sec^{-1}\left(\frac{x}{5}\right) + \frac{5\sqrt{x^2-25}}{x^2}\right] + C}$$

Don't have to rat'lize denoms.

Helps w/ 1, 3
$\int \csc\theta\, d\theta$
10: $\int \sec\theta\, d\theta$

Ex (#18) $\displaystyle\int \frac{1}{x\sqrt{25x^2+16}}\,dx$ ~~$u = 25x^2+16$~~?

Integrand has $\sqrt{(5x)^2+(4)^2}$

sin, tan, sec
Not x=

Sub: $5x = 4\tan\theta$
$x = \frac{4}{5}\tan\theta$
$dx = \frac{4}{5}\sec^2\theta\, d\theta$

$$= \int \frac{1}{(\frac{4}{5}\tan\theta)\sqrt{(4\tan\theta)^2+16}} \cdot \frac{4}{5}\sec^2\theta \, d\theta$$

can skip:
$$= \sqrt{16\tan^2\theta+16}$$
$$= 4\sqrt{\tan^2\theta+1}$$
$$= 4\sqrt{\sec^2\theta} \quad \text{(by Pyth. ID)}$$
$$= 4\sec\theta \quad \text{(ignore ||)}$$

OK: $\int \frac{du}{u}$ OK

$$= \int \frac{\sec^2\theta \, d\theta}{\tan\theta \cdot 4\sec\theta}$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan\theta} \, d\theta$$

What's a strategy for dealing w/ mixtures of sec, tan, ...

$$= \frac{1}{4}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}} \, d\theta$$

$$= \frac{1}{4}\int \csc\theta \, d\theta$$

Know Pop Quiz!

$$= \frac{1}{4}\ln|\csc\theta - \cot\theta| + C$$

Go back to x!!

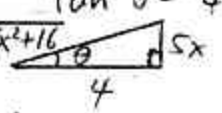
$5x = 4\tan\theta$

$\tan\theta = \frac{5x}{4} \Rightarrow \cot\theta = \frac{4}{5x}$

What's trig of $\csc\theta$?

$\csc\theta = ?$ In 1st S

$\tan\theta = \frac{5x}{4} \Rightarrow$



(or Pyth. Thm.)

$\sin\theta = \frac{5x}{\sqrt{25x^2+16}}$

$\csc\theta = \frac{\sqrt{25x^2+16}}{5x}$

$$= \boxed{\frac{1}{4}\ln\left|\frac{\sqrt{25x^2+16}}{5x} - \frac{4}{5x}\right| + C} \quad \text{or} \quad \ln\sqrt[4]{\left|\frac{\sqrt{5x}-1}{\sqrt{5x}+1}\right|} + C$$

9.4: PARTIAL FRACTION DECOMPOSITIONS (PFDs)

(A) $\int \frac{1}{1-x^2}\,dx$

Ex $\int_{-0.4}^{0.7} \frac{1}{1-x^2}\,dx$

sin values

$x = \sin\theta$ OK

SKIP

Let $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

(in $(-\frac{\pi}{2}, \frac{\pi}{2})$)

$dx = \cos\theta\,d\theta$

What does this integrand boil down to?

$$= \int_{\sin^{-1}(-0.4)}^{\sin^{-1}(0.7)} \underbrace{\frac{1}{1-\sin^2\theta}}_{=\sec\theta} \cos\theta\,d\theta$$

$$= \left[\ln|\sec\theta + \tan\theta|\right]_{\sin^{-1}(-0.4)}^{\sin^{-1}(0.7)}$$

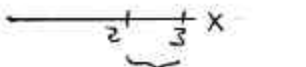
$$\vdots$$

$$\approx 1.29095$$

Ex $\int_2^3 \frac{1}{1-x^2}\,dx$

We're stuck! Or Are we?

~~$x = \sin\theta$~~



not sin values

Ex Find an expression for $\int \frac{1}{1-x^2}\,dx$ that will work ↑

$$= \int \underbrace{\frac{1}{(1+x)(1-x)}}_{\text{factor}}\,dx$$

Find a pfd.

Stewart 497: proper rat'l = Σ (form $\frac{A}{(ax+b)^r}$) (form $\frac{Ax+B}{(quad)^r}$)

$$\frac{1}{(1+x)(1-x)} = \underbrace{\frac{A}{1+x} + \frac{B}{1-x}}_{\text{Form of pfd}}$$

Distinct linear factors

LCD

Find A, B

· thru by $(1+x)(1-x)$

Who's missing from denom.?

Who's missing?

$$1 = A(1-x) + B(1+x) \quad \leftarrow \text{Basic Eq.}$$

Method 1 (Expand)

$$1 = A - Ax + B + Bx$$

Combine like terms.

$$1 = (-Ax + Bx) + (A+B)$$

$$1 = (-A+B)x + (A+B)$$

$1 = 0x+1 \Rightarrow$ set $=0$, set $=1$ ← Equate coeffs. of like powers.

$$\begin{cases} -A+B = 0 \\ A+B = 1 \end{cases}$$

$$2B = 1$$

$$B = \frac{1}{2}$$

$$\Rightarrow A = \frac{1}{2}$$

Method 2 (Pick convenient values of x,)

Basic Eq.: $1 = A(1-x) + B(1+x)$ (holds for all real x)

$x=1$ makes this 0; $x=-1$

Plug in $x=1$: $1 = A(0) + B(2) \Rightarrow B = \frac{1}{2}$

$x=-1$: $1 = A(2) + B(0) \Rightarrow A = \frac{1}{2}$

Stewart 448 cases where original eq. invalid: #67 F, G, Q polys, $\frac{F(x)}{Q(x)} = \frac{G(x)}{Q(x)}$ ∀x except where $Q(x)=0$ ⇒ Use continuity to prove $F(x)=G(x)$ ∀x

$F(a) = \lim_{x\to a} F(x) = \lim_{x\to a} G(x) = G(a)$

pfd: $\frac{1}{(1+x)(1-x)} = \frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{2}}{1-x}$

$$\int \frac{1}{(1+x)(1-x)}\,dx = \frac{1}{2}\left(\underbrace{\int \frac{1}{1+x}\,dx}_{\substack{u_1 = 1+x \\ du_1 = dx}} + \underbrace{\int \frac{1}{1-x}\,dx}_{\substack{u_2 = 1-x \\ du_2 = -dx}} \right)$$

$$= \frac{1}{2}\left(\int \frac{du_1}{u_1} - \int \frac{du_2}{u_2} \right)$$

$$= \frac{1}{2}\left(\ln|u_1| - \ln|u_2|\right) + K \quad \text{(In case you've used C)}$$

$$= \boxed{\frac{1}{2}\left(\ln|1+x| - \ln|1-x|\right) + K}$$

$$\boxed{\text{or } \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + K}$$

$$\boxed{\text{or } \ln\sqrt{\left|\frac{1+x}{1-x}\right|} + K}$$

Guess-and-✓ OK

Sols manual: +K

Do 1, 7, 9, NOT 5

Diff of logs = log(quotient)! Can ✓

Ex $\int_2^3 \frac{1}{1-x^2}\,dx$

$$= \left[\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right]_2^3$$

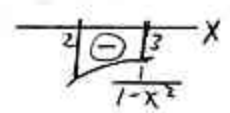
$$= \frac{1}{2}\left(\ln\left|\frac{1+3}{1-3}\right| - \ln\left|\frac{1+2}{1-2}\right|\right)$$

$$= \frac{1}{2}\left(\ln\left|\frac{4}{-2}\right| - \ln\left|\frac{3}{-1}\right|\right)$$

$$= \frac{1}{2}(\ln 2 - \ln 3) \text{ or } \ln\sqrt{\tfrac{2}{3}}$$

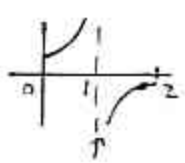
$$\approx -0.202733$$

Why is this < 0?



(Ex $\int_{-0.4}^{0.7} \frac{1}{1-x^2}\,dx \approx 1.29095$ (again))

Ex $\int_0^2 \frac{1}{1-x^2}\,dx$



Ch. 10

Can we use the same technique?

(B) Exs

Ex $\int \frac{-x^2-3x+32}{x^3-8x^2+16x}\,dx$ ← degree 2. ← 3

2 < 3, so proper
If improper, use long ÷.

Factor denom.
$x(x^2-8x+16)$
$x(x-4)^2$

pfd Form: $\frac{-x^2-3x+32}{x(x-4)^2} = \frac{A}{x} + \frac{B}{x-4} + \frac{C}{(x-4)^2}$

x: linear factor
$(x-4)(x-4)$: repeated linear factors
run up to the power

Find A, B, C

LCD

· thru by $x(x-4)^2$

Who's missing from denom.?

$-x^2-3x+32 = A(x-4)^2 + Bx(x-4) + Cx$

Like a cryptogram

Method 1 (Expand, combine like terms)

$-x^2-3x+32 = \underbrace{(A+B)}_{=-1}x^2 + \underbrace{(-8A-4B+C)}_{=-3}x + \underbrace{(16A)}_{=32}$

$16A = 32 \Rightarrow A = 2$

$A+B=-1 \Rightarrow 2+B=-1 \Rightarrow B=-3$

$-8A-4B+C=-3 \Rightarrow -16+12+C=-3 \Rightarrow C=1$

I skipped a lot of steps, but this is the overall schematic.

Method 2 (Plug into x)

$$-x^2-3x+32 = A(x-4)^2 + Bx(x-4) + Cx$$

$x=4$: $-16-12+32 = 0 + 0 + 4C$

$4 = 4C$

$\boxed{C=1}$

$x=0$: $-0-0+32 = A(0-4)^2 + 0 + 0$

$32 = 16A$

$\boxed{A=2}$

$$\underbrace{-x^2-3x+32}_{-x^2\ldots} = \underbrace{2(x-4)^2}_{2x^2\ldots} + \underbrace{\overset{?}{B}x(x-4)}_{Bx^2\ldots} + x$$

Equate x^2 terms:

(If $x \neq 0$ (arbitrary) $\Rightarrow \div x^2$)

$-x^2 = 2x^2 + Bx^2$

$-1 = 2 + B$

$\boxed{B=-3}$

or Plug in $x=1$, say.

.5?

$$\text{pfd: } \frac{-x^2-3x+32}{x(x-4)^2} = \frac{2}{x} + \frac{-3}{x-4} + \frac{1}{(x-4)^2}$$

$$\int \frac{-x^2-3x+32}{x(x-4)^2}\,dx = 2\int \frac{1}{x}\,dx - 3\underbrace{\int \frac{1}{x-4}\,dx}_{\substack{u=x-4 \\ \text{or Guess-and-}\checkmark}} + \underbrace{\int \frac{1}{(x-4)^2}\,dx}_{\substack{u=x-4 \\ du=dx}}$$

(Even if "u"s diff., don't have to write u_1, u_2 (irritating))

$$= 2\ln|x| - 3\ln|x-4| + \underbrace{\int u^{-2}\,du}_{\substack{= \frac{u^{-1}}{-1} + K \\ = -\frac{1}{x-4} + K}}$$

$$= \boxed{2\ln|x| - 3\ln|x-4| - \frac{1}{x-4} + K}$$

or $\ln \frac{x^2}{|x-4|^3} - \frac{1}{x-4} + K$

Up to 11

Ex $\int \frac{2x^4 - 4x^3 + 40x^2 - 71x + 82}{x^3 - 2x^2 + 16x - 32}\,dx$ ← deg 4, ← deg 3

$4 \geq 3$, so improper

Need for 16

Long ÷

$$\begin{array}{r} 2x \\ x^3 - 2x^2 + 16x - 32 \;\overline{)\; 2x^4 - 4x^3 + 40x^2 - 71x + 82} \\ \underline{-2x^4 + 4x^3 - 32x^2 + 64x} \\ 8x^2 - 7x + 82 \end{array}$$

subtract (flip signs)

← deg 2

2 < 3 ✓

Now, add (combine like terms):

I hope you know this...

Recall: $4\overline{)9}$ R1 $\Rightarrow 2\frac{1}{4}$ ← remainder, ← divisor

$$\int \left(2x + \frac{8x^2 - 7x + 82}{x^3 - 2x^2 + 16x - 32}\right) dx$$

Need for 17

Factor denom. by grouping

$(x^3 - 2x^2) + (16x - 32)$

$x^2(x-2) + 16(x-2)$

$(x^2 + 16)(x - 2)$

Otherwise, use $\pm \frac{p}{q}$, synthetic ÷.

Rational Roots Thm, Rat'l Zeros Test

$$\int \left(2x + \frac{8x^2 - 7x + 82}{(x^2+16)(x-2)}\right) dx$$

(Rat'l roots have form $\pm \frac{p}{q}$ ← factor of const term, ← factor of leading coeff.)

Find pfd.

ptd form: $\dfrac{8x^2-7x+82}{(x^2+16)(x-2)} = \dfrac{Ax+B}{x^2+16} + \dfrac{C}{x-2}$

x^2+16: irreducible quadratic factor (no real zeros); $x-2$: linear

Can't factor further over reals → Can't kill C easily

Find A, B, C

- thru LCD

$8x^2-7x+82 = (Ax+B)(x-2)+C(x^2+16)$

Hybrid of 2 methods

Plug in $x=2$: $32-14+82 = 0 + 20C$

$100 = 20C$

$C=5$

$8x^2-7x+82 = (Ax+B)(x-2)+5(x^2+16)$

Expand
Combine like terms

$= (A+5)x^2 + (-2A+B)x + (-2B+80)$

$A+5=8$ → $A=3$

$-2B+80=82$ → $B=-1$

ptd: $\dfrac{8x^2-7x+82}{(x^2+16)(x-2)} = \dfrac{3x-1}{x^2+16} + \dfrac{5}{x-2}$

$$\int \left(2x + \frac{8x^2-7x+82}{(x^2+16)(x-2)}\right) dx = \int 2x\,dx + \int \frac{3x-1}{x^2+16}\,dx + 5\int \frac{1}{x-2}\,dx$$

split

~~$u = x^2+16$~~

$$= x^2 + \underbrace{\int \frac{3x}{x^2+16}\,dx}_{①} - \underbrace{\int \frac{1}{x^2+16}\,dx}_{②} + 5\ln|x-2|$$

① $3\int \frac{x}{x^2+16}\,dx$ $\quad u = x^2+16$, $du = 2x\,dx$

$= \frac{3}{2}\int \frac{2x}{x^2+16}\,dx$ $\quad$ or $x\,dx = \frac{1}{2}du$

$= \frac{3}{2}\int \frac{du}{u}$

$= \frac{3}{2}\ln|u| + C_1$

$= \frac{3}{2}\ln(x^2+16) + C_1$ (x^2+16 always ≥ 0)

$= \frac{3}{2}\ln(x^2+16) + C_1$

Shortcut p. 437 formula in book easy to forget!

② $\int \frac{1}{x^2+16}\,dx$ $\quad x = 4\tan\theta$, $dx = 4\sec^2\theta\,d\theta$

$= \int \frac{1}{16\tan^2\theta + 16} \cdot 4\sec^2\theta\,d\theta$ $\quad 16(\tan^2\theta+1) = 16\sec^2\theta$

$= \frac{1}{16}\int \frac{1}{\sec^2\theta} \cdot 4\sec^2\theta\,d\theta$

$= \frac{1}{4}\int d\theta$

$= \frac{1}{4}\theta + C_2$

Find θ:

$x = 4\tan\theta$

$\frac{x}{4} = \tan\theta$

$\theta = \tan^{-1}\left(\frac{x}{4}\right)$

$= \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C_2$ (or p. 437 Shortcuts)

$\int \frac{1}{x^2+16}\,dx = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)+C$

$\int \frac{1}{x\sqrt{x^2-16}}\,dx = \frac{1}{4}\sec^{-1}\left(\frac{x}{4}\right)+C$

$\int \frac{1}{\sqrt{16-x^2}}\,dx = \sin^{-1}\left(\frac{x}{4}\right)+C$ (no $\frac{1}{4}$)

$$\boxed{x^2 + \tfrac{3}{2}\ln(x^2+16) - \tfrac{1}{4}\tan^{-1}\left(\tfrac{x}{4}\right) + 5\ln|x-2| + K}$$

Recap Long ÷ ⇒ poly + $\frac{N(x)}{D(x)}$ proper

Factor denom.

pfd form for

Find $A, B, C \Rightarrow$ get pfd

$\int$ term-by-term

$+K$ at end

Up to 19

Ex $\int \frac{x^3}{(x^2+9)^2}\,dx$ ← deg 3 ← deg 4 proper

pfd form: $\underbrace{\frac{x^3}{(x^2+9)^2}}_{\text{repeated irred. quadratic factors } (x^2+9)(x^2+9)} = \frac{Ax+B}{x^2+9} + \frac{Cx+D}{(x^2+9)^2}$

Just like w/ repeated linear factors, run up to the power

Find A, B, C, D

· thru by $(x^2+9)^2$

$x^3 = \underbrace{(Ax+B)(x^2+9) + (Cx+D)}_{\text{Expand}}$

$$x^3 = Ax^3 + 9Ax + Bx^2 + 9B + Cx + D$$
$$x^3 = (A)x^3 + (B)x^2 + \underbrace{(9A + C)}_{=0}x + \underbrace{(9B + D)}_{=0}$$

$A = 1$ $\quad$ $B = 0$ $\quad$ $9(1) + C = 0$ $\quad$ $9(0) + D = 0$

$C = -9$ $\quad$ $D = 0$

ptd: $\dfrac{x^3}{(x^2+9)^2} = \dfrac{x}{x^2+9} + \dfrac{-9x}{(x^2+9)^2}$

$$\int \frac{x^3}{(x^2+9)^2}\,dx = \int \frac{x}{x^2+9}\,dx - 9\int \frac{x}{(x^2+9)^2}\,dx$$

$u = x^2 + 9$

$du = 2x\,dx$

can skip in class

$$= \frac{1}{2}\int \frac{2x}{x^2+9}\,dx - \frac{9}{2}\int \frac{2x}{(x^2+9)^2}\,dx$$

$$= \frac{1}{2}\int \frac{du}{u} - \frac{9}{2}\int \underbrace{\frac{du}{u^2}}_{u^{-2}\,du}$$

$$= \frac{1}{2}\ln|u| - \frac{9}{2}\left(\frac{u^{-1}}{-1}\right) + K$$

$$= \frac{1}{2}\ln|\underbrace{x^2+9}_{\text{always} \geq 0}| + \frac{9}{2}\left(\frac{1}{x^2+9}\right) + K$$

$$= \boxed{\frac{1}{2}\ln(x^2+9) + \frac{9}{2x^2+18} + K}$$

© Big Picture

(Assume all coeffs. are real.)

We can ∫ any rational expression (in x, say)

$$\int \frac{N(x)}{D(x)}\,dx \qquad N(x), D(x) \text{ polys in } x,\ D(x) \neq 0$$

BUT we're limited by our ability to factor polys ($D(x)$)

What if $D(x)$ is an ugly 10th-deg poly!

Maybe use: Factor & cancel
u-sub
Long ÷ if $\deg(N(x)) \geq \deg(D(x))$

Thm $D(x)$ must be:

① a non-0 constant ⇒ ∫ poly Easy!

② linear (⇒ ln)

③ irreducible, quadratic (⇒ $\int \frac{1}{1+x^2}dx$ → tan⁻¹, $\int \frac{x}{1+x^2}dx$ → u-sub, ln (9.5))

or ④ some product of the above (incl. powers)
Use ptd!

Assume $\frac{N(x)}{D(x)}$ proper

Stewart 497
proper rat'l = Σ form $\frac{A}{(ax+b)^n}$, form $\frac{Ax+B}{(ax^2+bx+c)^n}$
always doable

#69: $\int \frac{1}{x^4+1}dx$

$x^4+1 = y^4+2x^2+1 - 2x^2$
$= (x^2+1)^2 - (\sqrt{2}x)^2$
Factor A^2-B^2

Ex Fair for test! $\int \frac{1}{x^2(x-1)^3(x^2+9)}\,dx$ (Find ptd form.)

$$= \underbrace{\frac{A}{x} + \frac{B}{x^2}}_{\text{run up}} + \underbrace{\frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}}_{\text{run up}} + \frac{Fx+G}{x^2+9}$$

9.5: DEALING w/ QUADRATICS

Ex (9.4) $\int \frac{1}{\underbrace{x^2+3x-10}_{\text{Factor}}}\,dx$ ~~u-sub~~

$= \int \underbrace{\frac{1}{(x+5)(x-2)}}\,dx$

$= \frac{A}{x+5} + \frac{B}{x-2}$

$\vdots$

$A = -\frac{1}{7},\ B = \frac{1}{7}$

$= \int \frac{-\frac{1}{7}}{x+5}\,dx + \int \frac{\frac{1}{7}}{x-2}\,dx$

Can use log props

$= \boxed{-\frac{1}{7}\ln|x+5| + \frac{1}{7}\ln|x-2| + C}$

Ex $\int \frac{x}{\underbrace{x^2+6x+10}_{\text{what if irred.?}}}\,dx$ ~~u-sub~~

Can't factor over reals

Complete the square (CTS):

$x^2+6x+10 = (x^2+6x+\textcircled{9})+10\textcircled{-9}$

$= (x+3)^2 + 1$

$= \int \frac{x}{(x+3)^2+1}\,dx$

Let $u = x+3 \xrightarrow{\text{Solve for } x} x = u-3$

$du = dx$

split

$$= \int \frac{u-3}{u^2+1}\,du$$

$$= \underbrace{\int \frac{u}{u^2+1}\,du}_{①} - \underbrace{\int \frac{3}{u^2+1}\,du}_{②}$$

u, v reek of parts

① $w = u^2+1$

$dw = 2u\,du \Rightarrow u\,du = \frac{1}{2}dw$

$$\frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + C_1$$

$$= \frac{1}{2}\ln(\underbrace{u^2+1}_{>0}) + C_1$$

$$= \frac{1}{2}\ln(u^2+1) + C_1$$

Go back to x!!

$$= \frac{1}{2}\ln[(x+3)^2+1] + C_1$$

should look familiar

$$= \frac{1}{2}\ln(x^2+6x+10) + C_1$$

② $3\tan^{-1}u + C_2$

$= 3\tan^{-1}(x+3) + C_2$

Q3

$$= \boxed{\frac{1}{2}\ln(x^2+6x+10) - 3\tan^{-1}(x+3) + C}$$

We can factor $10x-x^2$, but...

Ex $\int \frac{1}{(10x-x^2)^{3/2}}\,dx$ ~~u-sub~~

↑ Uh oh!

$\sqrt{}$ ⇒ probably no PFD

$$= \int \frac{1}{(\sqrt{10x-x^2})^3}\,dx$$

(Aim for trig sub.)

CTS: $10x-x^2 = -x^2+10x$

$= -(x^2-10x+25)+25$

(really, -25)

$= -(x-5)^2+25$

$= 25-(x-5)^2$

$$= \int \frac{1}{(\sqrt{25-(x-5)^2})^3}\,dx$$

Form: $\sqrt{a^2-u^2}$

sin, tan, or sec!

$u = a\sin\theta$

Sub: $x-5 = 5\sin\theta$

Can skip →

$(x = 5\sin\theta+5)$

$dx = 5\cos\theta\,d\theta$

$$= \int \frac{1}{(\sqrt{25-(5\sin\theta)^2})^3}\cdot 5\cos\theta\,d\theta$$

$= \sqrt{25-25\sin^2\theta}$

$= 5\sqrt{1-\sin^2\theta}$

$= 5\cos\theta$

$$= \int \frac{1}{(5\cos\theta)^3} \cdot 5\cos\theta \, d\theta$$

$$= \int \frac{1}{(5\cos\theta)^2} \, d\theta$$

$$= \int \frac{1}{25\cos^2\theta} \, d\theta$$

$$= \frac{1}{25} \int \sec^2\theta \, d\theta$$

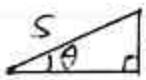
What's deriv. is $\sec^2\theta$?

$$= \frac{1}{25}\tan\theta + C$$

$\tan\theta = ?$

$x - 5 = 5\sin\theta$

$\sin\theta = \frac{x-5}{5}$



$(\sqrt{25-(x-5)^2}$, or)

$\sqrt{10x-x^2}$

Looks familiar?

Don't have to rationalize denom.

$$\tan\theta = \frac{x-5}{\sqrt{10x-x^2}}$$

$$= \boxed{\frac{1}{25}\left(\frac{x-5}{\sqrt{10x-x^2}}\right) + C}$$

9.6: RATIONALIZING SUBS

Ex $\int \frac{x-2}{\sqrt{4x+1}}\,dx$

Not really a rat'lizing sub. Like 5th 2 for Ex 1 in book.

Method 1

Let $u = 4x+1$ $\xrightarrow{\text{Solve for } x}$ $x = \frac{u-1}{4}$

$du = 4\,dx \Longrightarrow dx = \frac{1}{4}\,du$

$$\int \frac{\frac{u-1}{4}-2}{\sqrt{u}} \cdot \frac{1}{4}\,du$$

$$= \frac{1}{4}\int \frac{4\left(\frac{u-1}{4}-2\right)}{4(\sqrt{u})}\,du$$

$$= \frac{1}{4}\int \frac{(u-1)-8}{4\sqrt{u}}\,du$$

$$= \frac{1}{16}\int \frac{u-9}{u^{1/2}}\,du$$

$$= \frac{1}{16}\int (u^{1/2} - 9u^{-1/2})\,du$$

$$= \frac{1}{16}\left[\frac{u^{3/2}}{3/2} - 9\left(\frac{u^{1/2}}{1/2}\right)\right] + C$$

$$= \frac{1}{16}\left[\frac{2}{3}u^{3/2} - 18u^{1/2}\right] + C$$

Can factor out $\sqrt{4x+1}$

$$= \boxed{\frac{1}{24}(4x+1)^{3/2} - \frac{9}{8}\sqrt{4x+1} + C}$$

Method 2 (Rat'lizing Sub)

$$\int \frac{x-2}{\sqrt{4x+1}}\,dx$$

Let $u=\sqrt{4x+1}$

$u^2 = 4x+1 \Rightarrow x=\frac{u^2-1}{4}$

$2u\,du = 4\,dx \Rightarrow dx = \frac{1}{2}u\,du$

$$\int \frac{\frac{u^2-1}{4}-2}{u}\cdot\frac{1}{2}u\,du$$

$$= \frac{1}{2}\int \underbrace{\left(\frac{u^2-1}{4}-2\right)}_{\text{rational in } u}\,du$$

In fact poly

Easier

$$= \frac{1}{2}\int \left(\frac{u^2-1}{4}-\frac{8}{4}\right)du$$

$$= \frac{1}{2}\int \frac{u^2-9}{4}\,du$$

$$= \frac{1}{8}\int (u^2-9)\,du$$

$$= \frac{1}{8}\left[\frac{u^3}{3}-9u\right]+C$$

$$= \frac{1}{24}u^3-\frac{9}{8}u+C$$

$$= \frac{1}{2}\left[\frac{1}{4}\int (u^2-1)\,du - \int 2\,du\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}\left(\frac{u^3}{3}-u\right)-2u\right]+C$$

$$= \frac{1}{24}u^3-\frac{1}{8}u-u+C$$

$$= \frac{1}{24}u^3-\frac{9}{8}u+C$$

More complex setup, but we avoid nasc exps

$$= \boxed{\frac{1}{24}\left(\sqrt{4x+1}\right)^3-\frac{9}{8}\sqrt{4x+1}+C}$$

or $(4x+1)^{3/2}$

Can do 1, 3, 11 not 7

Ex $\int \frac{\sin(2x)}{3+\sin x}\,dx$

How before ⇒

Let $u = 3+\sin x \implies \sin x = u-3$

$du = \cos x\,dx$

(How do I pull $\cos x$ out of $\sin(2x)$?)

$$\int \frac{2\sin x\cos x}{3+\sin x}\,dx$$

$$= 2\int \frac{(u-3)\,du}{u}$$

$$= 2\int \left(1-\frac{3}{u}\right)du$$

$$= 2[u - 3\ln|u|] + C$$

$$= 2[(3+\sin x) - 3\ln|\underbrace{3+\sin x}_{>0}|] + C$$

C absorbs 6

Can do all but 7, 29

$$= \boxed{2\sin x - 6\ln(3+\sin x) + C}$$

Edwards 9.8 #7

Ex $\int \frac{1-\sqrt{x}}{1+\sqrt[4]{x}}\,dx$

Can factor right numerator or "conjugates" twice

How is $x^{1/2}$ related to $x^{1/4}$?

$\sqrt{x} = x^{1/2}$

$\sqrt[4]{x} = x^{1/4}$

LCD = 4

Let $u = x^{1/4} \Rightarrow u^4 = x$

$4u^3\,du = dx$

$u^2 = x^{1/2}$

Can I simp this frac?

$$\int \frac{1-u^2}{1+u} \cdot 4u^3\,du$$

$$= \int \frac{(1+u)(1-u)}{1+u} \cdot 4u^3\,du$$

$$= 4\int (u^3 - u^4)\,du$$

$$= 4\left[\frac{u^4}{4} - \frac{u^5}{5}\right] + C$$

$$= u^4 - \frac{4}{5}u^5 + C$$

$$= x - \frac{4}{5}(x^{1/4})^5 + C$$

Can do all but 29

$$= \boxed{x - \frac{4}{5}x^{5/4} + C}$$

Ex (#29) $\int \underbrace{\frac{1}{1+\sin x + \cos x}}_{\text{rational in } \sin x, \cos x}\,dx$

~~standard u-sub~~

Weierstrass Stewart 504-5 #55

Sub: $\sin x = \frac{2u}{1+u^2}$, $\cos x = \frac{1-u^2}{1+u^2}$, $dx = \frac{2}{1+u^2}\,du$ } I'll give as needed. $\Rightarrow \int (\text{rat'l in } u)\,du$

where $u = \tan\left(\frac{x}{2}\right)$ } Know.

MISI
CH. 9: REVIEW

$\int f(x)\,dx$

MISO

① Basic?

$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$ (POP QUIZ!)

$\int \frac{1}{2x+1}\,dx = \frac{1}{2}\ln|2x+1| + C$

Guess-and-✓
Use sparingly!

② Basic Manips.

$\int \frac{x+1}{x^3}\,dx$ ← split

Trig: → sin, cos
IBs (like Pyth.)

③ u-sub

→ x at end
(9.6) u = [esp. linear] $g(x)$ or $\sqrt[n]{g(x)}$, $u = x^{\frac{1}{LCD}}$
Solve for x (or x^n).

MISI (I May Mix Up the Order on the Test!)

④ ∫ by Parts

$\int \underbrace{(\text{Easy to D}_x)}_{u}\underbrace{(\text{Easy to }\int)}_{dv}$ $\quad du \cdot \int \cdot v$

$\int \underbrace{\ln x}_{u}\underbrace{dx}_{dv}$ ← ∫ (elementary func.) - what else would you do?

mead

I may forbid this on other problems, even though it may work well! See the web article (pp. 2-3).

Repeat? Reduction formulas. TURBO METHOD!

If $I = \sim\sim + \frac{1}{3} I$

$\frac{2}{3} I = \sim\sim + C$

$I = \frac{3}{2}(\sim\sim) + C$

⑤ Trig $\int$s

MISO (→ sin, cos; IDs ; u-sub)

Want: $\int \sin^{m} x \cos^{n} x \, dx$ (1 odd)

If m, n even ⇒ PRIs:

$\sin^2 x = \frac{1-\cos(2x)}{2}$

$\cos^2 x = \quad +$

$\int \tan^{m} x \sec^{n} x \, dx$ (odd or even) ⇒ Good! Guy! but

cot csc

Let u = "other guy"

$du = \underbrace{\sim\sim}_{(-1)\text{ peel}} dx$

Use Pyth. ID to make "u"s.

$\int \tan^{\text{even}} x \, dx$: Peel $\tan^2 x$, Pyth ID

$\int \sec^{\text{odd}} x \, dx$: Parts

⑥ Trig Subs

$\sqrt{9-x^2}$ $\quad x = 3\sin\theta$ $\quad$ Don't forget! $dx = \ldots$ $\quad$ (Really, $\theta = \sin^{-1}(\frac{x}{3})$, $|x| \le 3$)

$\sqrt{9-\underbrace{4x^2}_{(2x)^2}}$ $\quad 2x = 3\sin\theta$

x^2+9 (or $\sqrt{\ }$) $\quad x = 3\tan\theta$

$\sqrt{x^2-9}$ $\quad x = 3\sec\theta$

Use Pyth. ID

$\to x$

$\sin^{-1}, \ldots$?

Missing side should be familiar!

⑦ PFDs

$\int \frac{N(x)}{D(x)}\,dx$ $\quad$ (Avoid PFDs if see "$\sqrt{\text{stuff in } x}$")

rat'l (maybe after a sub?) $\quad$ Try ⑥ or ⑧?

Improper $\Rightarrow$ Long $\div$

$\frac{N(x)}{\text{Factored } D(x)} = \underbrace{\text{pfd form}}_{\text{Fair question!}}$ $\quad$ ($\frac{A}{x-1}$, $\frac{Bx+C}{x^2+16}$, run up to $(\ )^n$)

$\cdot$ thru $\quad$ irred. quadratic

Find $A, B, \ldots$:

Expand or

Pick x-values (Do 1st, then ↑?)

+K at end

⑧ CTS

$\int \frac{1}{\sqrt{\underbrace{x^2+6x+10}_{\text{CTS}}}}\,dx$ $\quad$ u- or Trig sub?

⑨ $\int$ (rat'l in $\sin x, \cos x$) dx

$u = \tan(\frac{x}{2})$

⑩ Table of $\int$s (A21) (9.7 - not tested)

Start on cheat sheet for final?