# 1.3: PREDICATES AND QUANTIFIERS

Ex 1 Let $P(x)$ denote the statement "$x < 5$."

$P$ is a predicate denoting the property "is less than 5."

$P(x)$ is not a proposition, but it becomes a prop. (T/F) once we assign a value to $x$.

$P(x)$ is a propositional function that depends on the variable $x$.

Ex In the case $x = 4$, "$4 < 5$" is T, so $P(4)$ is T.

Ex In the case $x = 5$, "$5 < 5$" is F, so $P(5)$ is F.

Ex 2 Let $Q(x,y)$ denote "$y = 3x + 1$."

Ex In the case $(x = 2, y = 7)$, "$7 = 3(2) + 1$" is T, so $Q(2,7)$ is T.

$\forall$ is the "universal quantifier" (say "for all")

$\forall x\, P(x)$ denotes the proposition

"$P(x)$ is T for all values of $x$ in the universe of discourse."

the domain of $P$ (we'll say "uod")

Some common "uod"s:

$\mathbb{Z}$ is the set of all integers.
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Z}^+$ is the set of all positive integers.
$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$

$\mathbb{R}$ is the set of all real #s.

Special case: Finite "uod"s

If the uod is a finite set, say $\{x_1, x_2, \ldots, x_n\}$, then

$$\underbrace{\forall x P(x)}_{\text{is T}} \underbrace{\iff}_{\text{exactly when}} \underbrace{P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)}_{\text{all these are T}}$$

Ex Let $P(x)$ be "$x$ is an integer."

If the uod $= \{1, 7, 15\}$, then $\forall x P(x)$ is T, because

$P(1) \wedge P(7) \wedge P(15)$ is T.

If the uod $= \{1, 7, \pi\}$, then $\forall x P(x)$ is F, because

$P(1) \wedge P(7) \wedge \underbrace{P(\pi)}$ is F.

one F ruins it for everybody

---

The uod can be an infinite set.

Ex Let $P(x)$ be "$2x > x$."

If the uod is $\mathbb{Z}^+$, $\forall x P(x)$ is T.

| $x$ | $2x$ |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 3 | 6 |
| ⋮ | ⋮ |

If the uod is $\mathbb{Z}$, $\forall x P(x)$ is F.
For example, $P(-1)$ is F. $(-2 \not> -1)$.
$x = -1$ is called a <u>counterexample</u>.

$\exists$ is the "existential quantifier" (say "there exists/is")

$\exists x\, P(x)$ denotes the proposition

"$P(x)$ is T for at least one element ($x$) in the uod."

Special case: Finite "uod"s

If the uod is $\{x_1, x_2, \ldots, x_n\}$, then

$$\underbrace{\exists x\, P(x)}_{\text{is T}} \underset{\text{exactly when}}{\iff} \underbrace{P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)}_{\text{at least one is T}}$$

Ex Let $P(x)$ be "$x$ is an integer."

If the uod $= \{1, \pi, e\}$, then $\exists x\, P(x)$ is T, because

$\underbrace{P(1)}_{\text{one T does it!}} \vee P(\pi) \vee P(e)$ is T.

If the uod $= \{\sqrt{2}, \pi, e\}$, then $\exists x P(x)$ is F, because

$\underbrace{P(\sqrt{2}) \vee P(\pi) \vee P(e)}_{\text{all are F}}$ is F.

Ex Let $P(x)$ be "$x^2 = 16$."
Let the uod be $\mathbb{Z}$.

Then, $\exists x\, P(x)$ is T.
For example, $P(4)$ is T. $P(-4)$, also.

Ex Let $Q(x)$ be "$x^2 = 17$."
Let the uod be $\mathbb{Z}$.

There is no integer whose square is 17,
so $\exists x\, Q(x)$ is F.

$\exists!$

Ex $P(x)$: "x is even", uod = $\mathbb{Z}$

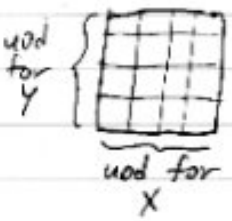
| | | | |
|---|---|---|---|
| $P(4)$ | "4 is even" | (T) | these are propositions |
| $\forall x\, P(x)$ | "all integers are even" | (F) | |
| $\exists x\, P(x)$ | "there is an even integer" | (T) | |

In these cases, x is <u>bound</u> by a value assignment or a quantifier.

In general, a propositional function becomes a proposition only when all the variables are bound.

# MORE THAN 1 VARIABLE

p. 31 - a great cure for insomnia!

Imagine a grid: 


① $\forall x \forall y\ P(x,y)$ is T $\Leftrightarrow$ all combos make P true


is F $\Leftrightarrow$ there is a counterexample that makes P false


Ex $P(x,y)$: "$xy = yx$"
uod = $\mathbb{R}$ for both $x, y$

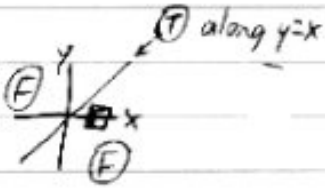
Then, $\forall x \forall y\ P(x,y)$ is T.

(Multiplication is commutative for all pairs of real #s.)

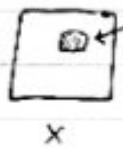
Ex $P(x,y)$: "$x - y = y - x$"
uod = $\mathbb{R}$ for both $x, y$

Then, $\forall x \forall y\ P(x,y)$ is F.

Counterexample: $x = 1, y = 0$
$1 - 0 \neq 0 - 1$

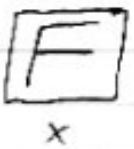


② $\exists x \exists y\ P(x,y)$ is T $\Leftrightarrow$




there is an example combo that makes P true

is F $\Leftrightarrow$




all combos make P false

Ex $P(x,y)$: "$x-y=y-x$"
uod = $\mathbb{R}$ for both $x, y$

Then, $\exists x \exists y\ P(x,y)$ is T.

For example, $(x=2, y=2)$:
$2-2=2-2$

Ex $P(x,y)$: "$xy=\pi$"
uod = $\mathbb{Z}$ for both $x, y$

Then, $\exists x \exists y\ P(x,y)$ is F.

(No two integers multiply to $\pi$.)

$\underbrace{\forall x\ \forall y}_{\text{can switch}}\ P(x,y) \Leftrightarrow \forall y \forall x\ P(x,y)$

$\underbrace{\exists x\ \exists y}_{\text{can switch}}\ P(x,y) \Leftrightarrow \exists y \exists x\ P(x,y)$

<u>BUT</u> order usually matters when mixing $\forall$s, $\exists$s.

③ $\forall x \exists y\ P(x,y)$ is T $\Leftrightarrow$

Each x can "find" a y that makes P true.

___ is F $\Leftrightarrow$

There is an x who can't find a y.

Ex $P(x,y)$: "$y - x = 6$"
uod = $\mathbb{R}$ for both x, y

Then, $\forall x \exists y\ P(x,y)$ is T. Why?

Regardless of which real # x is...

$y - x = 6$
$y = 6 + x$ (real #)

we can let y equal $6+x$, and P will be T.

Key: y can depend on x !!

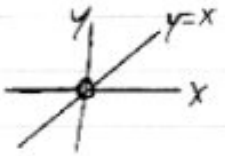
Idea: $P(0,6)$ is T
$P(1,7)$ is T
⋮
$P(x, 6+x)$ is T
↑ ⋮
any real #

Ex $P(x,y)$: "$\frac{x}{y}=1$"
uod = $\mathbb{R}$ for both $x,y$

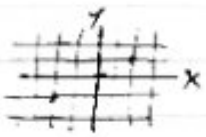
Then, $\forall x \exists y\, P(x,y)$ is F. Why?

$x=0$ can't "find" a $y$ (real #) to make $P$ true.

Note: Graph of $\frac{x}{y}=1$ (like our "grid")




Ex $P(x,y)$: "$\frac{x}{2}=y$"
uod = $\mathbb{Z}$ for both $x,y$




Then, $\forall x \exists y\, P(x,y)$ is F. Why?

If $x$ is odd, it can't find an integer $y$ to make $P$ true.

Key: Watch your "uod"s !!

If uod = $\mathbb{R}$ for both $x,y$, then T:
any real $x$ can pick $y$ to be $\frac{x}{2}$ (real).

How color?

④ $\exists x \forall y\ P(x,y)$ is T $\Leftrightarrow$

There is a "magic" $x$ that will make $P$ true, regardless of $y$ (in $y$'s uod)

is F $\Leftrightarrow$ there is no such column (Each $x$ can find a $y$ that makes $P$ false.)

Ex $P(x,y)$: "$\frac{\ln x}{y} = 0$"
uod = $\mathbb{R}^+$ for both $x, y$

Then, $\exists x \underbrace{\forall y \neq 0}$ $P(x,y)$ is T. Why?

exclude $y=0$ from consideration

$x=1$ is a "magic" real # that always makes $P$ true.

Ex $P(x,y)$: "$x+y=3$"
uod = $\mathbb{R}$ for both $x, y$

Then, $\exists x \forall y\ P(x,y)$ is F. Why!

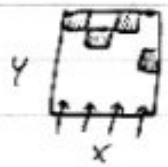
There is no "magic $x$" that works with <u>all</u> real "$y$"s to make $P$ true.

Order doesn't matter: $(x=3) \wedge (y=4)$

Order usually matters!

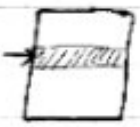
(Prop. A) $\forall x \exists y \; P(x,y)$

This is T $\Leftrightarrow$




(Prop. B) $\exists y \forall x \; P(x,y)$

This is T $\Leftrightarrow$

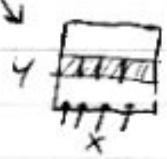



There is a magic y...

Note

If (Prop. A) is T, then (Prop. B) is not necessarily T.

If (Prop. B) is T, then (Prop. A) must be T.




Shorthand: $A \not\rightarrow B$
$B \rightarrow A$

## Ex (3 vars)

$P(x, y, z)$: "$xy = z$"
uod = $\mathbb{R}$ for $x, y$, and $z$

Then, $\forall x \forall y \exists z \; P(x, y, z)$ is T. Why?

Each pair of real #s has a real product.

$\exists x \exists z \forall y$ is T ($x=0, z=0$)

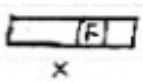
Also, $\exists z \forall x \forall y \; P(x, y, z)$ is F. Why?

There is no "magic" real # $z$ that is the product for every pair of real #s.

(See also Ex 24 on p. 32)

## NEGATIONS

$$\neg \forall x \, P(x) \iff \exists x \, \underbrace{\neg P(x)}_{\text{is T} \Leftrightarrow P(x) \text{ is F}}$$

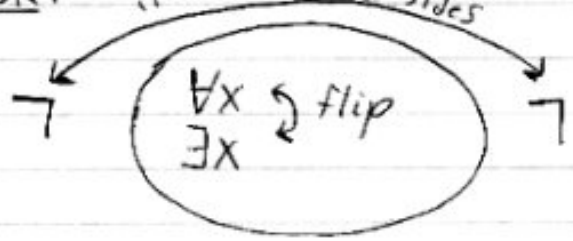



There is an $x$ that makes $P$ false

$$\neg \exists x \, P(x) \iff \forall x \, \neg P(x)$$

All $x$'s make $P$ false.

Trick: if $\neg$ switches sides



## HW Tips

In 1.3 (book), skip Exs. 12, 13, 16-21.

#10) Let $Q(x,y)$ be "$x$ has been a contestant on $y$"
uod for $x$ = the set of all students at your school.
uod for $y$ = the set of all quiz shows on TV.
Express in terms of $Q$, quantifiers, and connectives:

a) There is a student at your school who has been a contestant on a TV quiz show.

$$\exists x \exists y\, Q(x,y)$$

b) No student at your school has ever been a contestant on a TV quiz show.

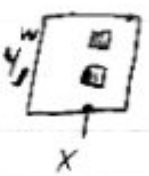
$$\forall x \forall y \neg Q(x,y)$$

or

$$\neg \exists x \exists y\, Q(x,y)$$

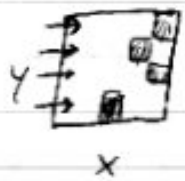
c) There is a student at your school who has been ... on Jeopardy and W.o.F.

$$\exists x \left(Q(x, \text{Jeopardy}) \wedge Q(x, \text{W.o.F.})\right)$$

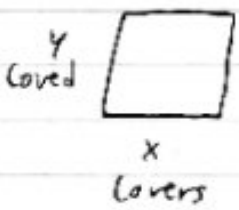


d) Every TV quiz show has had a student from your school...

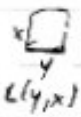
$\forall y \exists x Q(x, y)$




#11) $L(x,y)$: "x loves y"




(x, y have same mod) all people in the world

d)-f): different possible answers (¬,  since same mod)



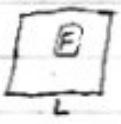
For me (Give ovals)

d) Nobody loves everybody

Book: $\forall x \exists y \neg L(x,y)$

Me: $\neg \exists x \forall y \; L(x,y)$

e) There is somebody whom Lydia does not love.



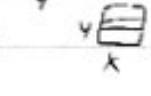

Book: $\exists x \neg L(lydia, x)$

Me: $\exists y \neg L(lydia, y)$

f) There is somebody whom no one loves.

Book: $\exists x \forall y \neg L(y,x)$

Me: $\exists y \forall x \neg L(x,y)$ or $\neg \forall y \exists x \; L(x,y)$




#22) TFTFTFTFTFFT